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BIPARTITE PSEUDO MV-ALGEBRAS

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Abstract

A bipartite pseudo MV-algebra A is a pseudo MV-algebra such that $A = M \cup M^{\sim}$ for some proper ideal M of A. This class of pseudo MV-algebras, denoted **BP**, is investigated. The class of pseudo MV-algebras A such that $A = M \cup M^{\sim}$ for all maximal ideals M of A, denoted $\mathbf{BP}_0$, is also studied and characterized.

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1. Preliminaries

In the theory of MV-algebras, the classes **BP** and $\mathbf{BP}_0$ are defined and studied by A. Di Nola, F. Liguori and S. Sessa in [3] and investigated by R. Ambrosio and A. Lettieri in [1]. Here we define and investigate the classes **BP** and $\mathbf{BP}_0$ of pseudo MV-algebras and we give some characterizations of them. Pseudo MV-algebras were introduced by G. Georgescu and A. Iorgulescu in [5] and later by J. Rachůnek in [6] (here called generalized MV-algebras or, in short, GMV-algebras) and they are a non-commutative generalization of MV-algebras.

Let $A = (A, \oplus, ^-, ^\sim, 0, 1)$ be an algebra of type $(2, 1, 1, 0, 0)$. Set $x \cdot y = (y^- \oplus x^-)^\sim$ for any $x, y \in A$. We consider that the operation $\cdot$ has priority to the operation $\oplus$, i.e., we will write $x \oplus y \cdot z$ instead of $x \oplus (y \cdot z)$. The algebra A is called a *pseudo MV-algebra* if for any $x, y, z \in A$ the following conditions are satisfied:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(A2) $x \oplus 0 = 0 \oplus x = x$;

(A3) $x \oplus 1 = 1 \oplus x = 1$;

(A4) $1^{\sim} = 0; 1^{-} = 0$;

(A5) $(x^{-} \oplus y^{-})^{\sim} = (x^{\sim} \oplus y^{\sim})^{-}$;

(A6) $x \oplus x^{\sim} \cdot y = y \oplus y^{\sim} \cdot x = x \cdot y^{-} \oplus y = y \cdot x^{-} \oplus x$;

(A7) $x \cdot (x^{-} \oplus y) = (x \oplus y^{\sim}) \cdot y$;

(A8) $(x^{-})^{\sim} = x$.

If the addition $\oplus$ is commutative, then both unary operations $^{-}$ and $^{\sim}$ coincide and then A is an MV-algebra.

Throughout this paper A will denote a pseudo MV-algebra. We will write $x^{\approx}$ instead of $(x^{\sim})^{\sim}$. For any $x \in A$ and $n = 0, 1, 2, \ldots$ we put

$$0x = 0 \text{ and } (n+1)\,x = nx \oplus x;$$

$$x^{0} = 1 \text{ and } x^{n+1} = x^{n} \cdot x.$$

Proposition 1.1 (Georgescu and Iorgulescu [5]). *The following properties hold for any $x, y \in A$:*

(a) $0^{-} = 1$;

(b) $1^{\approx} = 1$;

(c) $(x^{\sim})^{-} = x$;

(d) $(x^{-})^{\approx} = x^{\sim}$;

(e) $(x \oplus y)^{-} = y^{-} \cdot x^{-}; (x \oplus y)^{\sim} = y^{\sim} \cdot x^{\sim}$;

(f) $(x \cdot y)^{-} = y^{-} \oplus x^{-}; (x \cdot y)^{\sim} = y^{\sim} \oplus x^{\sim}$;

(g) $(x \oplus y)^{\approx} = x^{\approx} \oplus y^{\approx}$.

We define

$$x \leqslant y \Longleftrightarrow x^- \oplus y = 1.$$

As it is shown in [5], $(A, \leqslant)$ is a lattice in which the join $x \vee y$ and the meet $x \wedge y$ of any two elements x and y are given by:

$$\begin{aligned} x \vee y &= x \oplus x^\sim \cdot y = x \cdot y^- \oplus y; \\ x \wedge y &= x \cdot \big(x^- \oplus y\big) = (x \oplus y^\sim) \cdot y. \end{aligned}$$

For every pseudo MV-algebra A we set $\mathcal{L}(A) = (A, \vee, \wedge, 0, 1)$.

Proposition 1.2 (Georgescu and Iorgulescu [5]). *Let $x, y \in A$. Then the following properties hold:*

(a) $x \leqslant y \Longleftrightarrow y^- \leqslant x^-$;

(b) $x \leqslant y \Longleftrightarrow y^\sim \leqslant x^\sim$.

Following [4], we can consider the set $\mathrm{Inf}(A) = \{x \in A : x^2 = 0\}$. We have the following proposition.

Proposition 1.3 (Dymek and Walendziak [4]). *For every $x \in A$, the following conditions are equivalent:*

(a) $x \in \mathrm{Inf}(A)$;

(b) $2x^- = 1$;

(c) $2x^\sim = 1$.

By Proposition 1.3, $\mathrm{Inf}(A) = \{x \in A : 2x^- = 1\} = \{x \in A : 2x^\sim = 1\}$. We also have the following simple proposition.

Proposition 1.4. *The following conditions are equivalent for every $x \in A$ and $n \in \mathbb{N}$:*

(a) $x^n = 0$;

(b) $nx^- = 1$;

(c) $nx^\sim = 1$.

Proof. (a) ⇒ (b): Let $x^n = 0$. Then, by Proposition 1.1, $nx^- = (x^n)^- = 0^- = 1$.

(b) ⇒ (c): Suppose that $nx^- = 1$. Hence, by Proposition 1.1, $1 = 1^{\approx} = (nx^-)^{\approx} = n\,(x^-)^{\approx} = nx^{\sim}$.

(c) ⇒ (a): Suppose that $nx^{\sim} = 1$. Applying Proposition 1.1, we obtain $0 = 1^- = (nx^{\sim})^- = \left[(x^{\sim})^-\right]^n = x^n$. ■

Let $\mathrm{N}(A) = \{x \in A : x^n = 0 \text{ for some } n \in \mathbb{N}\}$. Elements of $\mathrm{N}(A)$ are called the *nilpotent* elements of A. From Proposition 1.4 we see that $\mathrm{N}(A) = \{x \in A : nx^- = 1 \text{ for some } n \in \mathbb{N}\} = \{x \in A : nx^{\sim} = 1 \text{ for some } n \in \mathbb{N}\}$. It is obvious that $\mathrm{Inf}(A) \subseteq \mathrm{N}(A)$.

Definition 1.5. A subset I of A is called an *ideal* of A if it satisfies the following conditions:

(I1) $0 \in I$;

(I2) *If* $x, y \in I$, then $x \oplus y \in I$;

(I3) *If* $x \in I$, $y \in A$ and $y \leqslant x$, then $y \in I$.

Under this definition, $\{0\}$ and A are the simplest examples of ideals.

Proposition 1.6 (Walendziak [8])**.** *Let I be a nonvoid subset of A. Then I is an ideal of A if and only if I satisfies conditions* (I2) *and*

(I3') *If $x \in I$, $y \in A$, then $x \wedge y \in I$.*

Denote by $\mathrm{Id}(A)$ the set of ideals of A and note that $\mathrm{Id}(A)$ ordered by set inclusion is a complete lattice.

Remark 1.7. Let $I \in \mathrm{Id}(A)$.

(a) *If* $x, y \in I$, then $x \cdot y, x \wedge y, x \vee y \in I$.

(b) I is an ideal of the lattice $\mathcal{L}(A)$.

For every subset $W \subseteq A$, the smallest ideal of A which contains W, i.e., the intersection of all ideals $I \supseteq W$, is said to be the ideal *generated* by W, and will be denoted $(W]$. For every $z \in A$, the ideal $(z] = (\{z\}]$ is called the *principal ideal generated by* z (see [5]), and we have

$$(z] = \{x \in A : x \leqslant nz \text{ for some } n \in \mathbb{N}\}.$$

Definition 1.8. Let I be a proper ideal of A (i.e., $I \neq A$).

(a) I is called *prime* if, for all $I_1, I_2 \in \mathrm{Id}(A)$, $I = I_1 \cap I_2$ implies $I = I_1$ or $I = I_2$.

(b) I is called *regular* if $I = \bigcap X$ implies that $I \in X$ for every subset X of $\mathrm{Id}(A)$.

(c) I is called *maximal* if whenever J is an ideal such that $I \subseteq J \subseteq A$, then either $J = I$ or $J = A$.

By definition, each regular ideal is prime.

Proposition 1.9 (Walendziak [8])**.** *If $I \in \mathrm{Id}(A)$ is maximal, then I is prime.*

Definition 1.10. A *cover* of a proper ideal I of A is a unique least ideal I^* which properly contains I.

Definition 1.11. A pseudo MV-algebra A is called *normal-valued* if for any regular ideal I of A and any $x \in I^*$, $x \oplus I = I \oplus x$.

An element $x \neq 0$ of a pseudo MV-algebra A is called *infinitesimal* (see [7]) if x satisfies condition

$$nx \leqslant x^- \text{ for each } n \in \mathbb{N}.$$

Proposition 1.12. *Let A be a pseudo MV-algebra and $x \in A$. Then the following conditions are equivalent:*

(a) *x is infinitesimal;*

(b) *$nx \leqslant x^\sim$ for each $n \in \mathbb{N}$;*

(c) *$x \leqslant (x^-)^n$ for each $n \in \mathbb{N}$;*

(d) *$x \leqslant (x^\sim)^n$ for each $n \in \mathbb{N}$.*

Proof. (a) $\Leftrightarrow$ (b): See Rachůnek [7].

(b) $\Rightarrow$ (c): Let $nx \leqslant x^\sim$ for each $n \in \mathbb{N}$. Then, by Propositions 1.2(a) and 1.1(e), $x = (x^\sim)^- \leqslant (nx)^- = (x^-)^n$ for each $n \in \mathbb{N}$.

(c) $\Rightarrow$ (b): Let $x \leqslant (x^-)^n$ for each $n \in \mathbb{N}$. Then, by Propositions 1.1(e) and 1.2(b), $nx = \left[(nx)^-\right]^\sim = \left[(x^-)^n\right]^\sim \leqslant x^\sim$ for each $n \in \mathbb{N}$.

(a) $\Leftrightarrow$ (d): Analogous. ■

Let us denote by Infinit(A) the set of all infinitesimal elements in A and by Rad(A) the intersection of all maximal ideals of A.

Proposition 1.13 (Rachůnek [7]). *Let A be a pseudo MV-algebra. Then:*

(a) $\mathrm{Rad}(A) \subseteq \mathrm{Infinit}(A)$.

(b) *If A is normal-valued, then* $\mathrm{Rad}(A) = \mathrm{Infinit}(A)$.

Proposition 1.14 (Dymek and Walendziak [4]). *Let A be a pseudo MV-algebra. Then* $\mathrm{Infinit}(A) \subseteq \mathrm{Inf}(A)$.

Proposition 1.15 (Dymek and Walendziak [4]). *Let A be a normal-valued pseudo MV-algebra. Then* $\mathrm{Inf}(A)$ *is an ideal of A if and only if* $\mathrm{Inf}(A) = \mathrm{Rad}(A)$.

2. Implicative ideals

Definition 2.1. An ideal I of A is called *implicative* if for any $x, y, z \in A$ it satisfies the following condition:

(Im) $(x \cdot y \cdot z \in I$ and $z^\sim \cdot y \in I) \Longrightarrow x \cdot y \in I$.

Proposition 2.2 (Walendziak [8]). *The implication* (Im) *is equivalent to*

(Im′) *For all $x, y, z \in A$, if $x \cdot y \cdot z^- \in I$ and $z \cdot y \in I$, then $x \cdot y \in I$.*

Proposition 2.3 (Walendziak [8]). *Let $I \in \mathrm{Id}(A)$. Then the following conditions are equivalent:*

(a) *I is implicative;*

(b) $\mathrm{N}(A) \subseteq I$;

(c) $\mathrm{Inf}(A) \subseteq I$.

Now we give an example of an ideal of a pseudo MV-algebra which is not implicative.

Example 2.4. Let A be the set of all increasing bijective functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$x \leqslant f(x) \leqslant x+1 \text{ for all } x \in \mathbb{R}.$$

Define the operations $\oplus, ^-, ^\sim$ and constans 0 and 1 as follows:

$$\begin{aligned}
(f \oplus g)(x) &= \min\{f(g(x)), x+1\}, \\
f^-(x) &= f^{-1}(x)+1, \\
f^\sim(x) &= f^{-1}(x+1), \\
0(x) &= x, \\
1(x) &= x+1.
\end{aligned}$$

Then $(A, \oplus, ^-, ^\sim, 0, 1)$ is a pseudo MV-algebra. Note that

$$\operatorname{Inf}(A) = \{f \in A : 2f^- = 1\} = \{f \in A : f(x) \leqslant f^{-1}(x)+1 \text{ for all } x \in \mathbb{R}\}$$

and the function $g(x) = x + \frac{1}{2}$ belongs to Inf(A). Observe that Inf(A) is not an ideal of A because $g \oplus g \notin \operatorname{Inf}(A)$. Now, define a function f as follows:

$$f(x) = \begin{cases} x+1 & \text{if } x \leqslant 0, \\ 1 + \dfrac{x}{2} & \text{if } 0 < x < 2, \\ x & \text{if } x \geqslant 2. \end{cases}$$

Obviously $f \in A$. Let I be the ideal generated by f^-, i.e.,

$$I = \{h \in A : h \leqslant nf^- \text{ for some } n \in \mathbb{N}\}.$$

Observe that $f^-(1) = 1$ and thus $nf^-(1) = 1$ for every $n \in \mathbb{N}$. Hence $g(1) = 1.5 > nf^-(1)$ for all n, i.e., $g \notin I$. Therefore $\operatorname{Inf}(A) \nsubseteq I$ and so, by Proposition 2.3, I is not an implicative ideal of A.

Proposition 2.5 (Walendziak [8]). *If* Inf(A) *is an ideal, then* Inf(A) *is implicative.*

Proposition 2.6. *If* $\mathrm{Inf}(A)$ *is an ideal of* A*, then* $\mathrm{Inf}(A) = \mathrm{N}(A)$.

Proof. Assume that $\mathrm{Inf}(A)$ is an ideal of A. Then, by Proposition 2.5, it is implicative. So, by Proposition 2.3, $\mathrm{N}(A) \subseteq \mathrm{Inf}(A)$ and since $\mathrm{Inf}(A) \subseteq \mathrm{N}(A)$, we obtain $\mathrm{Inf}(A) = \mathrm{N}(A)$. ■

For a nonvoid subset B of a pseudo MV-algebra A we put:

$$B^- = \{x^- : x \in B\} \text{ and } B^\sim = \{x^\sim : x \in B\}.$$

Proposition 2.7. *Let* I *be a proper ideal of* A *such that* $I^- = I^\sim$ *and let* A_I *be a subalgebra of* A *generated by* I*. Then* $A_I = I \cup I^- = I \cup I^\sim$.

Proof. First, it is clear that $I \cup I^- = I \cup I^\sim$. Now, we prove that $I \cup I^-$ is a subalgebra of A. Since $0 \in I$, we have $1 = 0^- \in I^- \subseteq I \cup I^-$. Thus $0, 1 \in I \cup I^-$.

Take arbitrary $x \in I \cup I^-$. Then $x \in I$ or $x \in I^-$. If $x \in I$, then $x^- \in I^-$ and therefore $x^- \in I \cup I^-$. If $x \in I^-$, then $x \in I^\sim$. This entails $x = y^\sim$ for some $y \in I$ and hence $x^- = y \in I$. Therefore $x^- \in I \cup I^-$ for any $x \in I \cup I^-$. Similarly, if $x \in I \cup I^-$, then $x^\sim \in I \cup I^\sim = I \cup I^-$.

Now, we show that $x \oplus y, x \cdot y \in I \cup I^-$ for every $x, y \in I \cup I^-$. We consider four cases.

Case 1. $x, y \in I$.
Since I is an ideal, $x \oplus y, x \cdot y \in I \subseteq I \cup I^-$.

Case 2. $x \in I, y \in I^-$.
Then, $x \cdot y \leqslant x$ and $x \in I$ entail $x \cdot y \in I \subseteq I \cup I^-$. Since $y \in I^-$, we have $y = z^-$, where $z \in I$ and hence, by Proposition 1.1(f), $x \oplus y = x \oplus z^- = (x^\sim)^- \oplus z^- = (z \cdot x^\sim)^- \in I^-$ because $z \cdot x^\sim \in I$. Thus $x \oplus y, x \cdot y \in I \cup I^-$.

Case 3. $x \in I^-, y \in I$.
Analogous.

Case 4. $x, y \in I^-$.
We have $x \oplus y = z^- \oplus t^- = (t \cdot z)^- \in I^-$ for some $t, z \in I$. Similarly, $x \cdot y = z^- \cdot t^- = (t \oplus z)^- \in I^-$. Therefore $x \oplus y, x \cdot y \in I \cup I^-$.

Finally, we get that $I \cup I^-$ is a subalgebra (containing I) of an algebra A and from this reason, $A_I \subseteq I \cup I^-$. It is obvious that $I \cup I^- \subseteq A_I$. ■

Remark 2.8. The assumption $I^- = I^\sim$ in Proposition 2.7 is necessary. Indeed, consider the pseudo MV-algebra A from Example 2.4. Take an ideal

$$I = \{h \in A : h \leqslant nf^- \text{ for some } n \in \mathbb{N}\}$$

generated by f^-, where

$$f(x) = \begin{cases} x+1 & \text{if } x \leqslant 0, \\ 1 + \dfrac{x}{2} & \text{if } 0 < x < 2, \\ x & \text{if } x \geqslant 2. \end{cases}$$

Thus $f \in I^\sim$. Since $f(1) = 1.5 > nf^-(1) = 1$ and $f^\sim(1) = 2 > nf^-(1)$, we have $f \notin I$ and $f^\sim \notin I$. Hence $f^- \notin I^-$ and $f \notin I^-$. Consequently we obtain $I^- \neq I^\sim$ and $f \notin I \cup I^-$, but $f \in A_I$.

Proposition 2.9 (Dymek and Walendziak [4]). *Let I be a prime ideal of A. Then the following conditions are equivalent:*

(a) *I is implicative*;

(b) $A = I \cup I^\sim (= I \cup I^-)$.

Proposition 2.10 (Dymek and Walendziak [4]). *Let I be a proper ideal of A. If $A = I \cup I^\sim (= I \cup I^-)$, then I is a maximal ideal of A generating A.*

Let us denote by $\mathrm{IRad}(A)$ the intersection of all implicative ideals of A. It is clear that $\mathrm{IRad}(A)$ is an implicative ideal of A, in fact, it is the smallest implicative ideal of A. By Propositions 1.13, 1.14 and 2.3, we have a ladder of inclusions:

(1) $$\mathrm{Rad}(A) \subseteq \mathrm{Infinit}(A) \subseteq \mathrm{Inf}(A) \subseteq \mathrm{N}(A) \subseteq \mathrm{IRad}(A).$$

Theorem 2.11. $(\mathrm{N}(A)] = \mathrm{IRad}(A)$.

Proof. Since $\mathrm{N}(A) \subseteq (\mathrm{N}(A)]$, it follows that $(\mathrm{N}(A)]$ is implicative. It is the smallest implicative ideal containing $\mathrm{N}(A)$ and hence the thesis. ■

Remark 2.12. We have also $(\mathrm{Inf}(A)] = \mathrm{IRad}(A)$ because $(\mathrm{Inf}(A)]$ is the smallest implicative ideal of A containing $\mathrm{Inf}(A)$.

Corollary 2.13. $\mathrm{Inf}(A)$ *is an ideal of* A *iff* $\mathrm{Inf}(A) = \mathrm{N}(A) = \mathrm{IRad}(A)$.

Theorem 2.14. $\mathrm{IRad}(A)$ *is a prime ideal of* A *iff* $A = \mathrm{IRad}\,(A) \cup (\mathrm{IRad}\,(A))^{\sim}$.

Proof. Let $\mathrm{IRad}(A)$ be a prime ideal of A. Since $\mathrm{IRad}(A)$ is implicative, we have, by Proposition 2.9, that $A = \mathrm{IRad}(A) \cup (\mathrm{IRad}\,(A))^{\sim}$.

If $A = \mathrm{IRad}(A) \cup (\mathrm{IRad}\,(A))^{\sim}$, then it is easy to see that $\mathrm{IRad}(A)$ is a maximal ideal of A. Hence, by Proposition 1.9, it is a prime ideal of A. ■

Corollary 2.15. $\mathrm{IRad}(A)$ *is a prime ideal of* A *iff* $A = \mathrm{IRad}\,(A) \cup (\mathrm{IRad}\,(A))^{-}$.

3. Bipartite pseudo MV-algebras

Now, we define the class **BP** of *bipartite* pseudo MV-algebras as follows: $A \in \mathbf{BP}$ iff $A = M \cup M^{\sim}$ for some proper ideal M of A. By Proposition 2.10, we have that if $A \in \mathbf{BP}$, then there is a maximal ideal of A generating A.

First, recall that a pseudo MV-algebra A is said to be *symmetric* if $x^{-} = x^{\sim}$ for any $x \in A$. It is shown in [2] that the variety of symmetric pseudo MV-algebras contains as a proper subvariety the variety of all MV-algebras. We have the following proposition.

Proposition 3.1. *Let* A *be a symmetric pseudo* MV*-algebra. Then* $A \in \mathbf{BP}$ *if and only if* A *is generated by some maximal ideal.*

Proof. Let A be a symmetric pseudo MV-algebra. If $A \in \mathbf{BP}$, then, by Proposition 2.10, there is a maximal ideal of A generating A.

Conversely, assume that A is generated by some maximal ideal M. Since A is symmetric, we have $M^{-} = M^{\sim}$. Hence, by Proposition 2.7, $A = M \cup M^{\sim}$. Therefore $A \in \mathbf{BP}$. ■

Proposition 3.2 (Dymek and Walendziak [4, Th. 3.5])**.** $A \notin \mathbf{BP}$ *iff* $(\mathrm{Inf}\,(A)] = A$.

Remark 3.3. Observe that for the pseudo MV-algebra A from Example 2.4, $(\mathrm{Inf}\,(A)] = A$. Thus, by Proposition 3.2, $A \notin \mathbf{BP}$.

Proposition 3.4. *If* $\mathrm{Inf}(A)$ *is a proper ideal of* A*, then* $A \in \mathbf{BP}$.

Proof. Assume that $\mathrm{Inf}(A)$ is a proper ideal of A. It is clear that there exists a maximal ideal M of A such that $\mathrm{Inf}(A) \subseteq M$. Then, by Proposition 2.3, M is implicative. From Proposition 2.9 we conclude that $A = M \cup M^{\sim}$. Thus $A \in \mathbf{BP}$. ■

Proposition 3.5. *$A \in \mathbf{BP}$ iff there exists an ideal I of A which is prime and implicative.*

Proof. Follows from Proposition 2.9. ∎

Theorem 3.6. *The class* **BP** *is closed under subalgebras.*

Proof. Let $A \in \mathbf{BP}$. Then there exists a proper ideal M of A such that $A = M \cup M^{\sim}$. Let B be a subalgebra of A. Then $I = M \cap B$ is a proper ideal of B. Since $(B \cap M)^{\sim} = B \cap M^{\sim}$, we have

$$\begin{aligned} B &= B \cap A = B \cap (M \cup M^{\sim}) = (B \cap M) \cup (B \cap M^{\sim}) \\ &= (B \cap M) \cup (B \cap M)^{\sim} = I \cup I^{\sim}. \end{aligned}$$

Therefore $B \in \mathbf{BP}$. ∎

Let A_t be a pseudo MV-algebra for $t \in T$ and let $A = \prod_{t \in T} A_t$ be the direct product of A_t. We can consider the canonical projection $\mathrm{pr}_t : A \to A_t$ which is, of course, a homomorphism of pseudo MV-algebras. If $t \in T$ and I_t is a proper ideal of A_t, then it is easily seen that $\mathrm{pr}_t^{-1}(I_t)$ is a proper ideal of A and that $\mathrm{pr}_t^{-1}\left(I_t^{-}\right) = \left[\mathrm{pr}_t^{-1}(I_t)\right]^{-}$ and $\mathrm{pr}_t^{-1}(I_t^{\sim}) = \left[\mathrm{pr}_t^{-1}(I_t)\right]^{\sim}$.

Theorem 3.7. *Let A and A_t for $t \in T$ be pseudo MV-algebras such that $A = \prod_{t \in T} A_t$. If $A_{t_0} \in \mathbf{BP}$ for some $t_0 \in T$, then $A \in \mathbf{BP}$.*

Proof. Since $A_{t_0} \in \mathbf{BP}$, we have $A_{t_0} = M_{t_0} \cup M_{t_0}^{\sim}$ for some proper ideal M_{t_0} of A_{t_0}. From the above discussion, $\mathrm{pr}_{t_0}^{-1}(M_{t_0})$ is a proper ideal of A and

$$\begin{aligned} A &= \mathrm{pr}_{t_0}^{-1}(A_{t_0}) = \mathrm{pr}_{t_0}^{-1}\left(M_{t_0} \cup M_{t_0}^{\sim}\right) = \mathrm{pr}_{t_0}^{-1}(M_{t_0}) \cup \mathrm{pr}_{t_0}^{-1}\left(M_{t_0}^{\sim}\right) \\ &= \mathrm{pr}_{t_0}^{-1}(M_{t_0}) \cup \left[\mathrm{pr}_{t_0}^{-1}(M_{t_0})\right]^{\sim}. \end{aligned}$$

Hence $A \in \mathbf{BP}$. ∎

Corollary 3.8. *The class* **BP** *is closed under direct products.*

Further, we define the class $\mathbf{BP}_0$ of pseudo MV-algebras as follows: $A \in \mathbf{BP}_0$ iff $A = M \cup M^{\sim}$ for all maximal ideals M of A. Note that if $A \in \mathbf{BP}_0$, then A is generated by all its maximal ideals. Remark that if A is a symmetric pseudo MV-algebra, then $A \in \mathbf{BP}_0$ if and only if A is generated by all its maximal ideals. Clearly, $\mathbf{BP}_0 \subseteq \mathbf{BP}$.

Theorem 3.9. $A \in \mathbf{BP}_0$ *iff* $\mathrm{Inf}(A) = \mathrm{Rad}(A)$.

Proof. Let $A \in \mathbf{BP}_0$. Then $A = M \cup M^{\sim}$ for every maximal ideal M of A. By Propositions 2.9 and 2.3, $\mathrm{Inf}(A) \subseteq M$ for every maximal ideal M of A and hence $\mathrm{Inf}(A) \subseteq \mathrm{Rad}(A)$. Thus, by (1), $\mathrm{Inf}(A) = \mathrm{Rad}(A)$.

Now, assume that $\mathrm{Inf}(A) = \mathrm{Rad}(A)$. Then $\mathrm{Inf}(A) \subseteq M$ for every maximal ideal M of A. By Propositions 2.3 and 2.9 we obtain that $A = M \cup M^{\sim}$ for every maximal ideal M of A. Thus $A \in \mathbf{BP}_0$. ∎

Corollary 3.10. *If* $A \in \mathbf{BP}_0$*, then* $\mathrm{Inf}(A) = \mathrm{N}(A)$.

Proof. From Theorem 3.9 we conclude that $\mathrm{Inf}(A)$ is an ideal of A. By Proposition 2.6, $\mathrm{Inf}(A) = \mathrm{N}(A)$. ∎

Corollary 3.11. $A \in \mathbf{BP}_0$ *iff* $\mathrm{Rad}(A)$ *is an implicative ideal of* A.

Proof. Let $A \in \mathbf{BP}_0$. Then, by Theorem 3.9, $\mathrm{Inf}(A) \subseteq \mathrm{Rad}(A)$ and hence, by Proposition 2.3, $\mathrm{Rad}(A)$ is an implicative ideal of A.

Conversely, assume that $\mathrm{Rad}(A)$ is an implicative ideal of A. Then, by Proposition 2.3, $\mathrm{Inf}(A) \subseteq \mathrm{Rad}(A)$ and thus, by (1), $\mathrm{Inf}(A) = \mathrm{Rad}(A)$. Therefore, by Theorem 3.9, $A \in \mathbf{BP}_0$. ∎

Theorem 3.12. *Let A be a pseudo MV-algebra. Then the following are equivalent:*

(a) $A \in \mathbf{BP}_0$;

(b) $\mathrm{Rad}(A) = \mathrm{Infinit}(A) = \mathrm{Inf}(A) = \mathrm{N}(A) = \mathrm{IRad}(A)$;

(c) *every maximal ideal of A is implicative.*

Proof. (a) ⇒ (b): Let $A \in \mathbf{BP}_0$. Then, by (1) and Theorem 3.9, $\mathrm{Rad}(A) = \mathrm{Infinit}(A) = \mathrm{Inf}(A)$. Hence $\mathrm{Inf}(A)$ is an ideal of A and by Corollary 2.13, $\mathrm{Inf}(A) = \mathrm{N}(A) = \mathrm{IRad}(A)$. Therefore (b) is true.

(b) ⇒ (c): Since $\mathrm{Inf}(A) = \mathrm{Rad}(A)$, $\mathrm{Inf}(A) \subseteq M$ for every maximal ideal M of A and by Proposition 2.3, every maximal ideal M of A is implicative.

(c) ⇒ (a): Since every maximal ideal M of A is implicative, we obtain by Proposition 2.9, $A = M \cup M^{\sim}$ for every maximal ideal M of A. Thus $A \in \mathbf{BP}_0$. ∎

Theorem 3.13. *Let A be a normal-valued pseudo MV-algebra. Then the following are equivalent:*

(a) $A \in \mathbf{BP}_0$;

(b) $\mathrm{Inf}(A)$ *is an ideal of* A;

(c) $\mathrm{Rad}(A) = \mathrm{Infinit}(A) = \mathrm{Inf}(A) = \mathrm{N}(A) = \mathrm{IRad}(A)$;

(d) *every maximal ideal of A is implicative.*

Proof. (a) $\Rightarrow$ (b): Follows from Theorem 3.9.

(b) $\Rightarrow$ (c): Follows from (1), Proposition 1.15 and Corollary 2.13.

(c) $\Rightarrow$ (d), (d) $\Rightarrow$ (a): Follow from Theorem 3.12. ■

From [2, Proposition 4.9], for any pseudo MV-algebras A, B we have:

$$\mathrm{Rad}\,(A \times B) = \mathrm{Rad}\,(A) \times \mathrm{Rad}\,(B)\,. \tag{2}$$

Lemma 3.14. *Let A, B be any pseudo MV-algebras. Then* $\mathrm{Inf}(A \times B) = \mathrm{Inf}(A) \times \mathrm{Inf}(B)$.

Proof. Let $(x, y) \in \mathrm{Inf}(A \times B)$. Then $(x, y)^2 = \left(x^2, y^2\right) = (0, 0)$ and hence $x^2 = y^2 = 0$. Thus $x \in \mathrm{Inf}(A)$ and $y \in \mathrm{Inf}(B)$, i.e., $(x, y) \in \mathrm{Inf}(A) \times \mathrm{Inf}(B)$.

Now, let $x \in \mathrm{Inf}(A)\,, y \in \mathrm{Inf}(B)$. Then $x^2 = y^2 = 0$. Hence $(x, y)^2 = (0, 0)$, i.e., $(x, y) \in \mathrm{Inf}(A \times B)$. Therefore $\mathrm{Inf}(A \times B) = \mathrm{Inf}(A) \times \mathrm{Inf}(B)$. ■

From (2), Lemma 3.14 and Theorem 3.9 we obtain the following theorem.

Theorem 3.15. *Let A, B be any pseudo MV-algebras. Then $A, B \in \mathbf{BP}_0$ iff $A \times B \in \mathbf{BP}_0$.*

We shall end the paper with two examples. The first one is an example of a pseudo MV-algebra which belongs to $\mathbf{BP}_0$, while the second one is an example of a pseudo MV-algebra which is in $\mathbf{BP}$ and is not in $\mathbf{BP}_0$.

Example 3.16 (Dymek and Walendziak [4]**).** Let $B = \{(1, y) : y \geqslant 0\} \cup \{(2, y) : y \leqslant 0\}$, $\mathbf{0} = (1, 0)$, $\mathbf{1} = (2, 0)$. For any $(a, b), (c, d) \in B$, we define operations $\oplus, ^-, ^\sim$ as follows:

$$(a,b) \oplus (c,d) = \begin{cases} (1, b+d) & \textit{if } a = c = 1, \\ (2, ad+b) & \textit{if } ac = 2 \textit{ and } ad + b \leqslant 0, \\ (2,0) & \textit{in other cases.} \end{cases}$$

$$(a,b)^- = \left(\frac{2}{a}, -\frac{2b}{a}\right),$$

$$(a,b)^\sim = \left(\frac{2}{a}, -\frac{b}{a}\right).$$

Then $B = (B, \oplus, ^-, ^\sim, \mathbf{0}, \mathbf{1})$ is a pseudo MV-algebra. Let $M = \{(1,y) : y \geqslant 0\}$. Then M is the unique maximal ideal of B and $B = M \cup M^\sim$ is generated by M. Thus $B \in \mathbf{BP}_0$ and so $B \in \mathbf{BP}$. Note that M is an implicative ideal of B and $\mathrm{Rad}(B) = \mathrm{Infinit}(B) = \mathrm{Inf}(B) = \mathrm{N}(B) = \mathrm{IRad}(B) = M$.

Example 3.17. Let A be the pseudo MV-algebra from Example 2.4 and B be the pseudo MV-algebra from Example 3.16. Since $B \in \mathbf{BP}$, we conclude, by Theorem 3.7, $A \times B \in \mathbf{BP}$. But, by Theorem 3.15, $A \times B \notin \mathbf{BP}_0$ because $A \notin \mathbf{BP}_0$.

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