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PRESOLID VARIETIES OF n-SEMIGROUPS

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Abstract

The class of all M-solid varieties of a given type τ forms a complete sublattice of the lattice $\mathcal{L}(\tau)$ of all varieties of algebras of type τ. This gives a tool for a better description of the lattice $\mathcal{L}(\tau)$ by characterization of complete sublattices. In particular, this was done for varieties of semigroups by L. Polák ([10]) as well as by Denecke and Koppitz ([4], [5]). Denecke and Hounnon characterized M-solid varieties of semirings ([3]) and M-solid varieties of groups were characterized by Koppitz ([9]). In the present paper we will do it for varieties of n-semigroups. An n-semigroup is an algebra of type (n), where the operation satisfies the $[i, j]$-associative laws for $1 \leq i < j \leq n$, introduced by Dörtnte ([2]). It is clear that the notion of a 2-semigroup is the same as the notion of a semigroup. Here we will consider the case $n \geq 3$.

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1. INTRODUCTION

Let τ be a fixed type of algebras, with fundamental operation symbols f_i of arity n_i, for $i \in I$. A hypersubstitution of type τ is a mapping which associates to every operation symbol f_i an n_i-ary term $\sigma(f_i)$ of type τ. Let $W_\tau(X)$ be the set of all terms of type τ on an alphabet $X := \{x_{1,}x_2, x_3, \ldots\}$. By $W_\tau(X_n)$ ($X_n := \{x_{1,}, \ldots, x_n\}$) we denote the set of all n-ary terms, $n \geq 1$. For $1 \leq m, n \in \mathbb{N}$ we define an operation $S^n_m : W_\tau(X_n) \times W_\tau(X_m)^n \to W_\tau(X_m)$ inductively as follows: For $(t_1, \ldots, t_n) \in W_\tau(X_m)^n$ we put:

(i) $S^n_m(x_i, t_1, \ldots, t_n) := t_i$ for $1 \leq i \leq n$;

(ii) $S^n_m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n) := f_i(S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_{n_i}, t_1, \ldots, t_n))$ for $i \in I, s_1, \ldots, s_{n_i} \in W_\tau(X_n)$ where $S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_{n_i}, t_1, \ldots, t_n)$ will be assumed to be already defined.

Any hypersubstitution σ can be uniquely extended to a mapping $\widehat{\sigma}$ on $W_\tau(X)$ inductively as follows:

(i) $\widehat{\sigma}[w] := w \text{for} w \in X$;

(ii) $\widehat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}_m(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$ for $i \in I, t_1, \ldots, t_{n_i} \in W_\tau(X_m)$ where $\widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}]$ will be assumed to be already defined.

A binary operation $\circ_h$ can be defined on the set $Hyp(\tau)$ of all hypersubstitutions of type τ, by letting $\sigma_1 \circ_h \sigma_2 = \widehat{\sigma}_1 \circ \sigma_2$, where $\circ$ is the usual composition of functions. The set $Hyp(\tau)$ is closed under this associative operation. It also contains an identity element for $\circ_h$, namely the identity hypersubstitution σ_{id} which maps every f_i to $f_i(x_1, \ldots, x_{n_i})$. Thus $Hyp(\tau)$ is a monoid.

Now let M be any submonoid of $Hyp(\tau)$. A variety V is called M-solid if for every $\sigma \in M$ and every identity $u \approx v$ in V, the identity $\widehat{\sigma}[u] \approx \widehat{\sigma}[v]$ holds in V. When M is the whole monoid $Hyp(\tau)$, an M-solid variety is called a solid variety. Two hypersubstitutions σ_1, σ_2 are said to be V-equivalent if for every operation symbol f_i of type τ, $\sigma_1(f_i) \approx \sigma_2(f_i)$

is an identity in V. In this case we write $\sigma_1 \sim_V \sigma_2$. In [11] it was proved that if $\widehat{\sigma}_1[s] \approx \widehat{\sigma}_1[t]$ is an identity in V for given terms $s, t \in W_\tau(X)$ and $\sigma_1 \sim_V \sigma_2$ then $\widehat{\sigma}_2[s] \approx \widehat{\sigma}_2[t]$ is an identity in V. Therefore, at most one element from each equivalence class of $\sim_V$ is needed to test the M-solidity.

The motivation of studying M-solid varieties comes from following result of Denecke and Reichel in [6]. For each monoid M of $Hyp(\tau)$, the collection of all M-solid varieties of type τ forms a complete lattice, which is a complete sublattice of the lattice $\mathcal{L}(\tau)$ of all varieties of type τ. This lattice $\mathcal{L}(\tau)$ is in general large and complicated, and difficult to study, and the M-solid sublattices give us a way to study at least some of its sublattices. Thus it may be useful to study the monoid $Hyp(\tau)$ and its submonoids M and the corresponding M-solid varieties, both in general and for specific type τ, and the intersection of the lattice of all M-solid varieties with a fixed variety of type τ. For specific types, much work has been done for type $\tau = (2)$, and in particular for varieties of semigroups. L. Polák ([10]) has given a characterization of the lattice of solid semigroup varieties, and various authors have studied M-solid semigroup varieties for various choices of M. Moreover, for type $\tau = (2, 2)$, in [3], all solid varieties of semirings are determined and, for type $\tau = (2, 1, 0)$, J. Koppitz ([9]) determined M-solid varieties of groups. More informations about hypersubstitutions, one can find in [8].
Our goal in this paper is a similar investigation for type (n), for $n \geq 3$. Only a few solid varieties of type (n) have been known (see [1] and [7]). We will consider the concept of an n-semigroup, which is a natural extension of the concept of a semigroup. An n-semigroup is an algebra of type (n), where the n-ary operation satisfies the $[i, j]$-associative laws

$$x_1 \dots x_{i-1}(x_i \dots x_{i+n-1})x_{i+n} \dots x_{2n-1} \approx$$

$$x_1 \dots x_{j-1}(x_j \dots x_{j+n-1})x_{j+n} \dots x_{2n-1}, \text{ for } 1 \leq i < j \leq n.$$

Each n-group is an n-semigroup (see Dörnte [2]). Each semigroup $(S; \cdot)$ induce an n-semigroup in the following way: Let $f_n : S^n \to S$ be defined by $f_n(a_1, a_2, \dots, a_n) := a_1 \cdot a_2 \cdot \dots \cdot a_n$ (we use the binary operation $\cdot$ of the given semigroup). Since $\cdot$ is associative, f_n satisfies the $[i, j]$-associative laws for $1 \leq i < j \leq n$, i.e., $(S; f_n)$ is an n-semigroup. Clearly, in the case $n = 2$ we have the $[1, 2]$-associative law $(x_1x_2)x_3 \approx x_1(x_2x_3)$. So the notion of a 2-semigroup is the same as the notion of a semigroup.

We also introduce the monoids $NPer(n)$ and $Pre(n)$ and give a characterization of all $NPer(n)$-solid as well as all $Pre(n)$-solid varieties of semigroups.

2. HYPERSUBSTITUTIONS OF TYPE (n)

In this section we present some background information about hypersubstitutions and varieties of type (n), and introduce the special monoids we shall be studying. We assume throughout a fixed type (n), with $n \geq 3$, so we have one n-ary operation symbol which we shall denote by f. For Σ any set of identities of type (n), we will denote by $Mod(\Sigma)$ the variety determined by the set Σ and by IdV we denote the set of all identities which hold in a given variety V. Because of the $[i,j]$-associative laws, $1 \leq i < j \leq n$, a term over a variety of n-semigroups can be regarded as a word of the length $(n-1)r+1$ for a suitable natural number r. By $l(t)$ we denote the length of a given term $t \in W_{(n)}(X)$ and $var(t)$ means the set of variables occurring in t. By $cv(t)$ we mean the cardinality of $var(t)$. For example, if $t = f(x_1, \ldots, x_1)$ then $l(t) = n$, $var(t) = \{x_1\}$, and $cv(t) = 1$. An identity $u \approx v$ is said to be normal if $u = v$ or both terms u and v are different from a variable. Since any hypersubstitution σ in $Hyp(n)$ is completely determined by what it does to f, we will denote by σ_t the hypersubstitution which maps f to the term t. For convenience, we list here some sets of terms and varieties of type (n) that we shall discuss later:

$W^{np}_{(n)}(X_n)$ be the set of all $t \in W_{(n)}(X_n)$ containing a subword s with $n = l(s) > cv(s)$;

$\widetilde{W}^{np}_{(n)}(X) := \{t \in W_{(n)}(X) \mid l(t) > cv(t)\}$;

$\widetilde{V}_n := Mod\{x_1 \ldots x_{2n-1} \approx x_1 \ldots x_{i-1} x_{i+1} x_{i+2} x_i x_{i+3} \ldots x_{2n-1} \mid 1 \leq i \leq 2n-3\}$;

$\widetilde{W}_n := Mod\{t \approx x^n \mid t \in W_{(n)}(X_n),\ n = l(t) > cv(t)\}$;

$V_n := \widetilde{V}_n \cap \widetilde{W}_n$.

It is easy to verify that there is no nontrivial solid variety of n-semigroups.

Theorem 1. *For each natural number $n \geq 3$ there is not nontrivial solid variety of n-semigroups.*

Proof. Let V be a solid variety of n-semigroups. Then $\widehat{\sigma}_{x_2}[(x_1 \ldots x_n) x_{n+1} \ldots x_{2n-1}] \approx \widehat{\sigma}_{x_2}[x_1 \ldots x_{n-1}(x_n \ldots x_{2n-1})] \in IdV$,i.e., $x_{n+1} \approx x_2 \in IdV$ and V is the trivial variety of type (n). ■

A hypersubstitution σ is called a pre-hypersubstitution if $\sigma(f)$ is not a variable. The set $Pre(n)$ of all pre-hypersubstitutions forms a submonoid of the monoid $Hyp(n)$ of all hypersubstitutions of type (n). A variety of n-semigroups is called presolid if it is M-solid for $M = Pre(n)$. Note that any solid variety is also presolid. By S_n we will denote the set of all bijections on the set $\{1, \ldots, n\}$. For $\pi \in S_n$, the hypersubstitution σ with $\sigma(f) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ will be denoted by σ_π. We will use the following notations of sets of hypersubstitutions:

$Pre(n) := Hyp(n) \setminus \{\sigma_{x_i} \mid 1 \leq i \leq n\}$ the set of all pre-hypersubstitutions;

$Per(n) := \{\sigma_\pi \mid \pi \in S_n\}$;

$Nper(n) := \{\sigma_t \mid t \in W^{np}_{(n)}(X_n)\} \cup \{\sigma_{id}\}$.

Proposition 2. *For $2 \leq n \in \mathbb{N}$, $Nper(n)$ forms a monoid.*

Proof. We have to check that $\sigma_1 \circ_h \sigma_2 \in Nper(n)$ for any $\sigma_1, \sigma_2 \in Nper(n)$. For this let $\sigma_1, \sigma_2 \in Nper(n)$. Then there are $r, t \in W^{np}_{(n)}(X_n)$ such that $\sigma_1(f) = r$ and $\sigma_2(f) = t$. In particular, r contains a subword s with $n = l(s) > cv(s)$. Further, $\widehat{\sigma}_1[t]$ contains a subterm $S^n_n(r, x_{i_1}, \ldots, x_{i_n})$. Since r contains a subword s with $n = l(s) > cv(s)$, the term $S^n_n(r, x_{i_1}, \ldots, x_{i_n})$ contains a subword $\widetilde{s}$ with $n = l(\widetilde{s}) > cv(\widetilde{s})$. Consequently, $\widehat{\sigma}_1[t]$ contains the subword $\widetilde{s}$ with $n = l(\widetilde{s}) > cv(\widetilde{s})$, i.e., $\sigma_1 \circ_h \sigma_2(f) = \widehat{\sigma}_1[t] \in W^{np}_{(n)}(X_n)$ and thus $\sigma_1 \circ_h \sigma_2 \in Nper(n)$. ■

3. Presolid varieties of n-semigroups

We begin the investigations of presolid varieties of n-semigroups by looking for a variety that contains all presolid varieties.

Proposition 3. *Let $3 \leq n \in \mathbb{N}$ and V be any $Pre(n)$-solid variety of n-semigroups. Then $V \subseteq \widetilde{V}_n$.*

Proof. Let $\pi \in S_n$ with $\pi(1) = 2$, $\pi(2) = 1$ and $\pi(k) = k$ for $3 \leq k \leq n$. If we apply σ_π to the $[1, n]$-associative law we get $x_{n+1}x_2x_1x_3 \ldots x_nx_{n+2} \ldots x_{2n-1} \approx x_2x_1x_3 \ldots x_{n+1}x_nx_{n+2}x_{n+3} \ldots x_{2n-1} \in IdV$ since V is $Pre(n)$-solid. By suitable substitution we get $x_1 \ldots x_{2n-1} \approx x_2 \ldots x_nx_1 x_{n+1} \ldots x_{2n-1} \in IdV$. If $n \geq 4$ then the application of σ_π to the $[3, 4]$-associative law gives $x_2x_1x_4x_3x_5 \ldots x_{2n-1} \approx x_2x_1x_3x_5x_4x_6 \ldots x_{2n-1} \in IdV$.

Both identities together provide $x_1 \dots x_{2n-1} \approx x_1 \dots x_{i-1}x_{i+1}x_{i+2} x_i x_{i+3} \dots x_{2n-1} \in IdV$ for $1 \leq i \leq n-2$. Let $\rho \in S_n$ with $\rho(2n-1) = 2n-2$, $\rho(2n-2) = 2n-1$ and $\rho(k) = k$ for $1 \leq k \leq 2n-3$. Dually, then the application of σ_ρ to the $[1,n]$-associative law as well as to the $[n-3, n-2]$-associative law (if $n \geq 4$) provides identities from which we can derive $x_1 \dots x_{2n-1} \approx x_1 \dots x_{i-1}x_{i+1}x_{i+2}x_i x_{i+3} \dots x_{2n-1} \in IdV$ for $n \leq i \leq 2n-3$. Finally, we have

$$x_1 \dots x_{2n-1}$$

$$\approx x_1 \dots x_{n-1}x_{n+1}x_{n+2}x_n x_{n+3} \dots x_{2n-1}$$

$$\approx x_1 \dots x_{n+1}x_{n-2}x_{n-1}x_{n+2}x_n x_{n+3} \dots x_{2n-1}$$

$$\approx x_1 \dots x_{n+1}x_{n-2}x_n x_{n-1}x_{n+2}x_{n+3} \dots x_{2n-1}$$

$$\approx x_1 \dots x_{n-2}x_n x_{n+1}x_{n-1}x_{n+2}x_{n+3} \dots x_{2n-1}, \text{ i.e.,}$$

$$x_1 \dots x_{2n-1} \approx x_1 \dots x_{n-2}x_n x_{n+1}x_{n-1}x_{n+2}x_{n+3} \dots x_{2n-1} \in \text{ IdV.}$$

Altogether we have $x_1 \dots x_{2n-1} \approx x_1 \dots x_{i-1}x_{i+1}x_{i+2}x_i x_{i+3} \dots x_{2n-1} \in IdV$ for $1 \leq i \leq 2n-3$. ■

Now we will determine identities satisfying by presolid varieties.

Lemma 4. *Let $4 \leq n \in 2\mathbb{N}$ and V be any $Pre(n)$-solid variety of n-semigroups. Then $x_1 \dots x_{2n-1} \approx x_{\pi(1)} \dots x_{\pi(2n-1)}$ for all $\pi \in S_{2n-1}$.*

Proof. Let $\pi \in S_{2n-1}$with $\pi(1) = 2$, $\pi(2) = 1$ and $\pi(k) = k$ for $3 \leq k \leq 2n-1$. If we apply σ_π to the $[1,n]$-associative law we get $x_{n+1}x_2x_1x_3 \dots x_n x_{n+2} \dots x_{2n-1} \approx x_2x_1x_3 \dots x_{n+1}x_n x_{n+2} \dots x_{2n-1} \in IdV$ since V is $Pre(n)$-solid and by suitable substitution we obtain

$$x_1 \dots x_{2n-1} \approx x_2 \dots x_n x_1 x_{n+1} \dots x_{2n-1} \in IdV. \tag{1}$$

By Proposition 3 we have $V \subseteq \widetilde{V}_n$. Using the identities of $\widetilde{V}_n$ we get $x_2 \dots x_n x_1 x_{n+1} \dots x_{2n-1} \approx x_2x_1x_3 \dots x_{2n-1} \in IdV$ (since n is a even number). Together with (1) we obtain $x_1 \dots x_{2n-1} \approx x_2x_1x_3 \dots x_{2n-1} \in IdV$. It is easy to see that one can derive $x_1 \dots x_{2n-1} \approx x_{\pi(1)} \dots x_{\pi(2n-1)}$ for all $\pi \in S_{2n-1}$ from $x_1 \dots x_{2n-1} \approx x_2x_1x_3 \dots x_{2n-1}$ and the identities of $\widetilde{V}_n$. ■

Lemma 5. *Let $3 \leq n \in \mathbb{N}$, $2n-1 \leq p \in (n-1)\mathbb{N}+1$ and V be a variety of n-semigroups with $V \subseteq \widetilde{V}_n$. Then for each $\pi \in S_p$ holds*

$$x_{\pi(1)} \dots x_{\pi(p)} \approx x_1 \dots x_p \in IdV \text{ or}$$

$$x_{\pi(1)} \dots x_{\pi(p)} \approx x_2 x_1 x_3 \dots x_p \in IdV.$$

Proof. Let $\pi \in S_p$. We consider the term $x_{\pi(1)} \dots x_{\pi(p)}$ and move step by step $x_p, x_{p-1}, \dots, x_3$ to the $p^{th}, (p-1)^{th}, \dots, 3^{th}$ position using the identities of $\widetilde{V}_n$. Then we have on the first both positions x_1x_2 or x_2x_1. This shows $x_{\pi(1)} \dots x_{\pi(p)} \approx x_1 \dots x_p \in IdV$ or $x_{\pi(1)} \dots x_{\pi(p)} \approx x_2x_1x_3 \dots x_p \in IdV$. ■

It is easy to check that $Nper(n) \subseteq Pre(n)$. So, any presolid variety has to be $Nper(n)$-solid. Next we find the lattice of all $Nper(n)$-solid varieties of n-semigroups.

Lemma 6. *Let $3 \leq n \in \mathbb{N}$ and V be any variety of n-semigroups with $V \subseteq V_n$. Then for each $t \in \widetilde{W}^{np}_{(n)}(X)$ holds $t \approx z^n \in IdV$.*

Proof. Let $t \in \widetilde{W}^{np}_{(n)}(X)$. Then there is a variable $w \in X$ that occurs at least two times in t. If $l(t) = n$ then $l(t) > cv(t)$ and $t \approx x^n \in IdV$ since $V \subseteq \widetilde{W}_n$. Suppose now that $l(t) > n$. Using the identities of $\widetilde{V}_n$ we can move w on the first and the second position, respectively, i.e., $t \approx wwu_3 \dots u_{l(t)}$ with $u_3, \dots, u_{l(t)} \in X$. Since $x_1x_1x_3 \dots x_n \approx z^n \in IdV$ we have $wwu_3 \dots u_{n-1}(u_n \dots u_{l(t)}) \approx z^n \in IdV$, i.e., $t \approx z^n \in IdV$. ■

Lemma 7. *Let $3 \leq n \in \mathbb{N}$ and V be any nontrivial variety of n-semigroups with $V \subseteq \widetilde{W}_n$. Then only normal identities hold in V.*

Proof. Assume that a non-normal identity $u \approx v$ holds in V. Then $u \neq v$ and one of the terms u and v is a variable. Without loss of generality let u be a variable. Since V is a nontrivial variety the term v ($\neq u$) is not a variable. Then by substitution we get $y \approx y^{l(v)} \in IdV$ where $l(v) > 1$. Clearly, $l(v) = r(n-1)+1$ for some natural number $r \geq 1$. From $xy^{n-1} \approx z^n \in IdV$ it follows $y^{r(n-1)+1} \approx z^n \in IdV$, i.e., $y^{l(v)} \approx z^n \in IdV$. But $y \approx y^{l(v)}$ and $y^{l(v)} \approx z^n$ provide $x \approx y$, and V is the trivial variety, a contradiction. ■

Proposition 8. *Let $3 \leq n \in \mathbb{N}$. A nontrivial variety V of n-semigroups is $Nper(n)$-solid iff $V \subseteq \widetilde{W}_n$.*

Proof. Assume that V is $Nper(n)$-solid. We have $t_1 := x_1x_2^{n-1} \in W_{(n)}^{np}(X_n)$, i.e., $\sigma_{t_1} \in Nper(n)$ and its application to the $[1,3]$-associative law gives

$$x_1x_2^{n-1}x_{n+1}^{n-1} \approx x_1x_2^{n-1} \in IdV. \tag{1}$$

Further, we have $t_2 := x_2x_3^{n-1} \in W_{(n)}^{np}(X_n)$, i.e., $\sigma_{t_2} \in Nper(n)$ and its application to the $[1,2]$-associative law gives

$$x_{n+1}x_{n+2}^{n-1} \approx x_3x_4^{n-1}x_{n+2}^{n-1} \in IdV. \tag{2}$$

Then one obtains $xy^{n-1} \overset{(1)}{\approx} xy^{n-1}z^{n-1} \overset{(2)}{\approx} wz^{n-1} \in IdV$, i.e., we have $xy^{n-1} \approx z^n \in IdV$. Dually, we can show that $x^{n-1}y \approx z^n \in IdV$. Let now $t \in W_{(n)}(X_n)$ with $n = l(t) > cv(t)$. Then there are $u_1, \ldots, u_n \in X$ such that $t = u_1 \ldots u_n$. Since $l(t) > cv(t)$ there are $i, j \in \{1, \ldots, n\}$ with $i < j$ such that $u_i = u_j$. Then the term $s := x_1 \ldots x_{j-1}x_ix_{j+1} \ldots x_n$ belongs to $W_{(n)}^{np}(X_n)$, i.e., $\sigma_s \in Nper(n)$. Without loss of generality let $i \neq 1$. Then the application of σ_s to the $[1,j]$-associative law gives $x_1 \ldots x_{j-1}x_ix_{j+1} \ldots x_nx_{n+1} \ldots x_{n+j-2}x_{n+i-1}x_{n+j} \ldots x_{2n-1} \approx x_1 \ldots x_{j-1}x_ix_{n+j} \ldots x_{2n-1}$. Then $x_{n+1} \notin \{x_1, \ldots, x_{j-1}, x_i, x_{n+j}, \ldots, x_{2n-1}\}$ since $1 < i < j \neq 1$. So, we substitute x_{n+1} by x_{n+1}^n and get $x_1 \ldots x_{j-1}x_ix_{n+j} \ldots x_{2n-1} \approx x_1 \ldots x_{j-1}x_ix_{j+1} \ldots x_nx_{n+1}^n \ldots x_{n+j-2}x_{n+i-1}x_{n+j} \ldots x_{2n-1}$. It is easy to check that one can derive $x_1 \ldots x_{j-1}x_ix_{j+1} \ldots x_nx_{n+1}^n \ldots x_{n+j-2}x_{n+i-1}x_{n+j} \ldots x_{2n-1} \approx z^n$ using $xy^{n-1} \approx x^{n-1}y \approx z^n \in IdV$, i.e., one gets $x_1 \ldots x_{j-1} x_ix_{n+j} \ldots x_{2n-1} \approx z^n \in IdV$. Consequently, if we substitute $x_i$by u_i for $1 \leq i \leq n$ we get $u_1 \ldots u_n \approx z^n \in IdV$, i.e., $t \approx z^n \in IdV$. Altogether, $V \subseteq \widetilde{W}_n$.

Suppose now that $V \subseteq \widetilde{W}_n$. Let $t \in W_{(n)}^{np}(X_n)$. Then t contains a subterm s with $n = l(s) > cv(s)$ and there are words u and v (the empty word λ is also possible for u as well as for v) such that $t = usv$. Since $s \approx z^n \in IdV$ we have $t \approx uz^nv \in IdV$. The repeated application of $xy^{n-1} \approx x^{n-1}y \approx z^n \in IdV$ to uz^nv gives finally $uz^nv \approx z^n$, i.e., $t \approx z^n \in IdV$. This shows that any $\sigma \in Nper(n)$ is V-equivalent to $\sigma_{x_1^n}$.

Let $u \approx v \in IdV$. If $u = v$ then clearly $\widehat{\sigma}_{x_1^n}[u] \approx \widehat{\sigma}_{x_1^n}[v] \in IdV$. If $u \neq v$ and $u \approx v$ is a normal identity of V then there are natural numbers $r, s \geq 1$ such that $\widehat{\sigma}_{x_1^n}[u] \approx u_1^{n^r}$ and $\widehat{\sigma}_{x_1^n}[v] \approx v_1^{n^s}$ where u_1 (v_1) is the first letter in u (in v). From $xy^{n-1} \approx z^n \in IdV$ it follows $x^n \approx z^n \in IdV$ and thus $u_1^{n^r} \approx v_1^{n^s} \in IdV$, i.e., $\widehat{\sigma}_{x_1^n}[u] \approx \widehat{\sigma}_{x_1^n}[v] \in IdV$. Since only normal identities are satisfied in V by Lemma 7 we can conclude that V is $Nper(n)$-solid. ■

After the following lemma we are able to characterize all presolid varieties of n-semigroups.

Lemma 9. *Let $3 \leq n \in 2\mathbb{N}+1$, V be a variety of n-semigroups with $V \subseteq \widetilde{V}_n$, and $\sigma \in Per(n)$. Then there holds*

$$\widehat{\sigma}[x_1 \dots x_i(x_{i+1} \dots x_{i+n})x_{i+n+1} \dots x_{2n-1}] \approx x_1 \dots x_{2n-1} \in IdV$$

for $0 \leq i \leq n-1$.

Proof. Let $\pi \in S_n$. Without loss of generality let $i = 0$. Then

(1) $x_{\pi(1)} \dots x_{\pi(n)} x_{n+1} \dots x_{2n-1} \approx x_1 \dots x_{2n-1} \in IdV$ or

(2) $x_{\pi(1)} \dots x_{\pi(n)} x_{n+1} \dots x_{2n-1} \approx x_2 x_1 x_3 \dots x_{2n-1} \in IdV$ by Lemma 5. We put $y_1 := x_1 \dots x_n$ in case (1) ($y_1 := x_2 x_1 x_3 \dots x_n$ in case (2)) and $y_j := x_{n+j-1}$ for $2 \leq j \leq n$. Using the identities of $\widetilde{V}_n$ it is easy to check that $y_{\pi(1)} \dots y_{\pi(n)} \approx x_1 \dots x_{2n-1} \in IdV$ in case (1) and $y_{\pi(1)} \dots y_{\pi(n)} \approx x_{n+1} x_2 x_1 x_3 \dots x_n x_{n+2} \dots x_{2n-1} \in IdV$ in case (2), respectively. Further, we have $x_{n+1} x_2 x_1 x_3 \dots x_n x_{n+2} \dots x_{2n-1} \approx x_1 x_{n+1} x_2 x_3 \dots x_n x_{n+2} \dots x_{2n-1}$ $\approx x_1 x_2 x_3 \dots x_n x_{n+1} x_{n+2} \dots x_{2n-1} \in IdV$ (since n is an odd number). This shows that $\widehat{\sigma}_\pi[(x_1 \dots x_n)x_{n+1} \dots x_{2n-1}] \approx S^n_{2n-1}(\sigma_\pi(f), S^n_{2n-1}(\sigma_\pi(f), x_1, \dots, x_n), x_{n+1}, \dots, x_{2n-1}) \approx x_1 \dots x_{2n-1} \in IdV$. ∎

Theorem 10. *Let $n \geq 3$ be a natural number and V be a nontrivial variety of n-semigroups. Then V is $Pre(n)$-solid iff the following statements hold:*

(i) *$V \subseteq V_n$;*

(ii) *If $x_{\pi(1)} \dots x_{\pi(n)} \approx x_1 \dots x_n \in IdV$ for some $\pi \in S_n$ then $x_{\pi\circ s(1)} \dots x_{\pi\circ s(n)} \approx x_{s(1)} \dots x_{s(n)} \in IdV$ for all $s \in S_n$;*

(iii) *If $n \in 2\mathbb{N}$ then $x_1 \dots x_{2n-1} \approx x_{\pi(1)} \dots x_{\pi(2n-1)}$ for all $\pi \in S_{2n-1}$.*

Proof. Suppose that V is $Pre(n)$-solid. Then $V \subseteq \widetilde{V}_n$ by Proposition 3. Further, V is $Nper(n)$-solid since $Nper(n) \subseteq Pre(n)$. Then by Proposition 8 we get $V \subseteq \widetilde{W}_n$. Therefore, $V \subseteq \widetilde{V}_n \cap \widetilde{W}_n = V_n$ and it holds (i). Suppose that $x_{\pi(1)} \dots x_{\pi(n)} \approx x_1 \dots x_n \in IdV$ for some $\pi \in S_n$. Further let $\rho \in S_n$. Then $\sigma_\rho \in Pre(n)$. Since V is $Pre(n)$-solid we have $\widehat{\sigma}_\rho[x_1 \dots x_n] \approx \widehat{\sigma}_\rho[x_{\pi(1)} \dots x_{\pi(n)}] \in IdV$, i.e., $x_{\pi\circ\rho(1)} \dots x_{\pi\circ\rho(n)} \approx x_{\rho(1)} \dots x_{\rho(n)} \in IdV$. This shows (ii). Finally, (iii) it follows from Lemma 4.

Suppose that (i)–(iii) are satisfied. Let $\sigma_t \in Pre(n)$. If $\sigma_t \notin Per(n)$ then $t \in \widetilde{W}^{np}_{(n)}(X)$ and $t \approx z^n \in IdV$ by Lemma 6, i.e., σ_t is V-equivalent to $\sigma_{x_1^n}$, where $\sigma_{x_1^n} \in Nper(n)$. But (i) implies that V is $Nper(n)$-solid by Proposition 8. Thus $\widehat{\sigma}_{x_1^n}[u] \approx \widehat{\sigma}_{x_1^n}[v] \in IdV$ for all $u \approx v \in IdV$, i.e., $\widehat{\sigma}_t[u] \approx \widehat{\sigma}_t[v] \in IdV$ for all $u \approx v \in IdV$. Let now $\sigma_t \in Per(n)$ and $u \approx v \in IdV$. If $var(u) \neq var(v)$ then without loss of generality there is a $w \in var(u)\backslash var(v)$. We substitute w by x^n and get $\widetilde{u} \approx v \in IdV$ from $u \approx v \in IdV$ where x^n is a subterm of $\widetilde{u}$, i.e., $\widetilde{u} \in \widetilde{W}^{np}_{(n)}(X)$. Then by Lemma 6 we have $\widetilde{u} \approx x^n \in IdV$, i.e., $u \approx v \approx x^n \in IdV$. If $l(u) > cv(u)$ or $l(v) > cv(v)$ then $u \in \widetilde{W}^{np}_{(n)}(X)$ or $v \in \widetilde{W}^{np}_{(n)}(X)$ and thus $u \approx v \approx x^n \in IdV$ by Lemma 6. Consequently, if $var(u) \neq var(v)$ or $l(u) > cv(u)$ or $l(v) > cv(v)$ then $u \approx v \approx x^n \in IdV$. If, in particular, $l(u) = cv(u)$ then $u = u_1 \dots u_{l(u)}$ with $u_1, \dots, u_{l(u)} \in X$ and there is a $\pi \in S_{l(u)}$ such that $\widehat{\sigma}_t[u] \approx u_{\pi(1)} \dots u_{\pi(l(u))}$. But from $u \approx x^n \in IdV$ we get by the substitution $u_i \mapsto u_{\pi(i)}$ for $1 \leq i \leq l(u)$ that $u_{\pi(1)} \dots u_{\pi(l(u))} \approx x^n \in IdV$, i.e., $\widehat{\sigma}_t[u] \approx x^n \in IdV$. If, in particular, $l(v) = cv(v)$ then we get $\widehat{\sigma}_t[v] \approx x^n \in IdV$ in the same matter. If $l(u) > cv(u)$ ($l(v) > cv(v)$) then $u \in \widetilde{W}^{np}_{(n)}(X)$ ($v \in \widetilde{W}^{np}_{(n)}(X)$) and it is easy to check that $\widehat{\sigma}_t[u] \in \widetilde{W}^{np}_{(n)}(X)$ ($\widehat{\sigma}_t[v] \in \widetilde{W}^{np}_{(n)}(X)$), too. Then $\widehat{\sigma}_t[u] \approx x^n \in IdV$ ($\widehat{\sigma}_t[v] \approx x^n \in IdV$) by Lemma 6. Consequently, $\widehat{\sigma}_t[u] \approx x^n \approx \widehat{\sigma}_t[v] \in IdV$. The remaining case is $var(u) = var(v)$ and $l(u) = cv(u)$ and $l(v) = cv(v)$. We put $s := l(u)$ and $\{u_1, \dots, u_s\} = var(u) = var(v)$. Because of Lemma 9 (if $n \in 2\mathbb{N} + 1$) and of (iii) (if $n \in 2\mathbb{N}$), respectively, we have $\widehat{\sigma}_t[x_1 \dots x_{i-1}(x_i \dots x_{i+n-1})x_{i+n} \dots x_{2n-1}] \approx \widehat{\sigma}_t[x_1 \dots x_{j-1}(x_j \dots x_{j+n-1})x_{j+n} \dots x_{2n-1}] \in IdV$ for $1 \leq i < j \leq n$. Therefore we can assume that

$$u = \big(\dots \big(u_1 \dots u_n\big) u_{n+1} \dots u_{2n-1}\big) \dots u_{s-1} u_s\big)$$

$$v = \big(\dots \big(u_{\pi(1)} \dots u_{\pi(n)}\big) u_{\pi(n+1)} \dots u_{\pi(2n-1)}\big) \dots u_{\pi(s-1)} u_{\pi(s)}\big)$$

for some permutation $\pi \in S_s$. Further there is a $\rho \in S_n$ such that $\sigma_t = \sigma_\rho$. If $s = 1$ we have obviously $\widehat{\sigma}_\rho[u] \approx \widehat{\sigma}_\rho[v] \in IdV$. If $s = n$ then $\widehat{\sigma}_\rho[u] \approx u_{\rho(1)} \dots u_{\rho(n)}$ and $\widehat{\sigma}_\rho[v] \approx u_{\pi\circ\rho(1)} \dots u_{\pi\circ\rho(n)}$. By (ii) from $x_{\pi(1)} \dots x_{\pi(n)} \approx x_1 \dots x_n \in IdV$ it follows $x_{\pi\circ\rho(1)} \dots x_{\pi\circ\rho(n)} \approx x_{\rho(1)} \dots x_{\rho(n)} \in IdV$, i.e., $\widehat{\sigma}_\rho[u] \approx \widehat{\sigma}_\rho[v] \in IdV$. Let now $s > n$. Then there is a $\phi \in S_s$ such that $\widehat{\sigma}_t[u] \approx u_{\phi(1)} \dots u_{\phi(s)}$ and $\widehat{\sigma}_t[v] \approx u_{\pi\circ\phi(1)} \dots u_{\pi\circ\phi(s)}$.

By Lemma 5 we have $\widehat{\sigma}_t[u] \approx u_1 \dots u_s$ or $\widehat{\sigma}_t[u] \approx u_2 u_1 u_3 \dots u_s =: \widetilde{u}$. If $\widehat{\sigma}_t[u] \approx u$, i.e., $x_{\phi(1)} \dots x_{\phi(s)} \approx u_1 \dots u_s \in IdV$ then by the substitution $u_i \mapsto u_{\pi(i)}$ for $1 \leq i \leq s$ we get $u_{\pi\circ\phi(1)} \dots u_{\pi\circ\phi(s)} \approx u_{\pi(1)} \dots u_{\pi(s)} \in IdV$, i.e., $\widehat{\sigma}_t[v] \approx v$, and from $u \approx v \in IdV$ it follows $\widehat{\sigma}_t[u] \approx \widehat{\sigma}_t[v] \in IdV$. If $\widehat{\sigma}_t[u] \approx \widetilde{u}$, i.e., $u_{\phi(1)} \dots u_{\phi(s)} \approx u_2 u_1 u_3 \dots u_s$ then by the same substitution we get $u_{\pi\circ\phi(1)} \dots u_{\pi\circ\phi(s)} \approx u_{\pi(2)} u_{\pi(1)} u_{\pi(3)} \dots u_{\pi(s)} =: \widetilde{v}$, i.e., $\widehat{\sigma}_t[v] \approx \widetilde{v} \in IdV$. Moreover, from Lemma 5 we get

$$u_{\pi(2)} u_{\pi(1)} u_{\pi(3)} \dots u_{\pi(s)} \approx u_1 \dots u_s \text{ or}$$

$$u_{\pi(2)} u_{\pi(1)} u_{\pi(3)} \dots u_{\pi(s)} \approx u_2 u_1 u_3 \dots u_s$$

as well as

$$u_{\pi^{-1}(2)} u_{\pi^{-1}(1)} u_{\pi^{-1}(3)} \dots u_{\pi^{-1}(s)} \approx u_1 \dots u_s \text{ or}$$

$$u_{\pi^{-1}(2)} u_{\pi^{-1}(1)} u_{\pi^{-1}(3)} \dots u_{\pi^{-1}(s)} \approx u_2 u_1 u_3 \dots u_s.$$

i.e.,

$$u_2 u_1 u_3 \dots u_s \approx u_{\pi(1)} \dots u_{\pi(s)} \text{ or}$$

$$u_2 u_1 u_3 \dots u_s \approx u_{\pi(2)} u_{\pi(1)} u_{\pi(3)} \dots u_{\pi(s)} .$$

This shows $\widetilde{v} \approx u$ or $\widetilde{v} \approx \widetilde{u}$ as well as $\widetilde{u} \approx v$ or $\widetilde{u} \approx \widetilde{v}$. This implies $\widetilde{v} \approx \widetilde{u}$ or both $\widetilde{v} \approx u$ and $\widetilde{u} \approx v$ hold in V. Since $u \approx v \in IdV$ we have altogether $\widetilde{v} \approx \widetilde{u} \in IdV$ and thus $\widehat{\sigma}_t[u] \approx \widehat{\sigma}_t[v] \in IdV$ because of $\widehat{\sigma}_t[u] \approx \widetilde{u} \in IdV$ and $\widehat{\sigma}_t[v] \approx \widetilde{v} \in IdV$. ■

Let us apply Theorem 10 for the case $n = 3$. We obtain the following characterization of all presolid varieties of 3-semigroups.

Corollary 11. *A nontrivial variety of* 3*-semigroups is* $Pre(3)$*-solid iff* $V \subseteq Mod\{(xyz)wt \approx x(yzw)t \approx xy(zwt) \approx yzxwt \approx xzwyt \approx xywtz, xyx \approx x^2y \approx xy^2 \approx z^3\} =: W$ *and it holds the following condition:*

(∗) *If* $x_1x_2x_3 \approx x_{\pi(1)}x_{\pi(2)}x_{\pi(3)} \in IdV$ *for some* $\pi \in \{(12),(13),(23)\}$

then $x_1x_2x_3 \approx x_{\rho(1)}x_{\rho(2)}x_{\rho(3)} \in IdV$ *for all* $\rho \in S_3$.

Proof. Suppose that V is $Pre(3)$-solid. Then the conditions (i) and (ii) of Theorem 10 are satisfied. From (i) it follows that $xyzwt \approx yzxwt \approx xzwyt \approx xywtz \in IdV$ and $xyx \approx x^2y \approx xy^2 \approx z^3 \in IdV$. Hence $V \subseteq W$. Using (ii) we can verify condition $(*)$: If $\pi = (13)$, i.e., $x_1x_2x_3 \approx x_3x_2x_1 \in IdV$ then $x_2x_1x_3 \approx x_2x_3x_1 \in IdV$ (for $s = (12)$). Both identities provide $x_1x_2x_3 \approx x_1x_3x_2 \approx x_2x_3x_1 \approx x_2x_1x_3 \approx x_2x_3x_1 \approx x_1x_3x_2 \in IdV$. If $\pi = (12)$, i.e., $x_1x_2x_3 \approx x_2x_1x_3 \in IdV$ then $x_1x_3x_2 \approx x_2x_3x_1 \in IdV$ (for $s = (23)$). If $\pi = (23)$, i.e., $x_1x_2x_3 \approx x_1x_3x_2 \in IdV$ then $x_2x_1x_3 \approx x_3x_1x_2 \in IdV$ (for $s = (12)$). In the latter two cases, we conclude in the same matter as before.

Suppose now that $V \subseteq W$ and $(*)$ is satisfied. Since $V \subseteq W$, the condition (i) of Theorem 10 holds. We have now to show that also condition (ii) is satisfied. For this let $\pi \in S_3$. If $\pi \in \{(1), (12), (13), (23)\}$ then the condition is satisfied by $(*)$. If $\pi = (123)$, i.e., $x_1x_2x_3 \approx x_2x_3x_1 \in IdV$ then we have to check that also $x_2x_1x_3 \approx x_3x_2x_1 \in IdV$, $x_3x_2x_1 \approx x_1x_3x_2 \in IdV$, $x_1x_3x_2 \approx x_2x_1x_3 \in IdV$, $x_2x_3x_1 \approx x_3x_1x_2 \in IdV$, and $x_3x_1x_2 \approx x_1x_2x_3 \in IdV$. Obviously, these five equations are consequences of the given identity $x_1x_2x_3 \approx x_2x_3x_1 \in IdV$. If $\pi = (132)$ the we conclude in the same matter. This shows (ii). Condition (iii) can be neglected since 3 is odd. Altogether, V is $Pre(3)$-solid by Theorem 10. ∎

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