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COMMUTATION OF OPERATIONS AND ITS RELATIONSHIP WITH MENGER AND MANN SUPERPOSITIONS

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Abstract

The article considers a problem from Trokhimenko paper [13] concerning the study of abstract properties of commutations of operations and their connection with the Menger and Mann superpositions. Namely, abstract characterizations of some classes of operation algebras, whose signature consists of arbitrary families of commutations of operations, Menger and Mann superpositions and their various connections are found. Some unsolved problems are given at the end of the article.

Keywords: Menger superposition, Superassociativity, (unitary) Menger algebra, selektor, n-ary groupoid, (extented) Menger multisemigroup (of operations), commutation of an operation, unar (of commutations), Mann superposition, abstract characterization of Menger algebras.

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1. Introduction

Let $\Gamma_n(Q)$ denote the set of all operations of the arity $n+1$ defined on an arbitrary fixed set Q. Let σ be a substitution (permutation) on the set $\{0, \ldots, n\}$, and let f be an arbitrary operation from $\Gamma_n(Q)$. Denote by σf an $(n+1)$-ary operation determined by the equality*

*the sign := means "the left side is equal to the right one by the definition".

(1) $$(\sigma f)(x_0,\dots,x_n) := f(x_{\sigma 0},\dots,x_{\sigma n})$$

for all $x_0, \dots, x_n \in Q$. The operation σf is said to be a *commutation* or, more precisely, the *σ-commutation* of the operation f. It is called the *principal parastrophe* of f in the quasigroup theory.

It is easy to see that every substitution σ of the set $\{0,\dots,n\}$ determines a unary operation on $\Gamma_n(Q)$, which we will denote by the same symbol σ. Different substitutions determine different operations. An algebra $(\Phi; U)$, where $\Phi \subseteq \Gamma_n(Q)$ and U is a subset of the symmetrical group S_{n+1}, is said to be a *unar of commutations of* $(n+1)$*-ary operations* or an *operation commutation unar.* K. Menger (in [10]) introduced the notion of superposition on the set $\Gamma_n(Q)$:

(2) $$(f[f_0,\dots,f_n])\,(a_0,\dots,a_n) := f\,(f_0(a_0,\dots,a_n),\dots,f_n(a_0,\dots,a_n))$$

for all $a_0, \dots, a_n \in Q$ Some authors (see, for instance, [11]) call it *Menger superposition* and the corresponding algebras $(\Phi; \mathcal{O})$ and $(\Phi; \mathcal{O}, U)$, where $\Phi \subseteq \Gamma_n(Q)$, $U \subseteq S_{n+1}$ and

$$\mathcal{O}(f, f_0,\dots,f_n) := f[f_0,\dots,f_n],$$

are called *Menger algebra* and *extended Menger algebra of* $(n+1)$*-ary operations* respectively. The abstract class of all Menger algebras of $(n+1)$-ary operations is described by the identity

(3) $$\big(x[y_0,\dots,y_n]\big)[z_0,\dots,z_n] = x\big[y_0[z_0,\dots,z_n],\dots,y_n[z_0,\dots,z_n]\big],$$

the so-called *superassociativity* ([11]).

A pair $(M; f)$ is called an *n-ary groupoid*, if M is a set and f is an n-ary operation defined on the set M. A superassociative $(n+1)$-ary groupoid is called a *Menger algebra of the rank n*.

Selectors or *projections* play a special role in the theory of operations. In most cases they are denoted by $e_0, \dots, e_n$ and are defined as follows

(4) $$e_i(a_0,\dots,a_{i-1},a_i,a_{i+1},\dots,a_n) := a_i \quad (i = 0,\dots,n),$$

for all $a_j \in Q$.

An abstract analogue of the selectors are the elements $e_0, \ldots, e_n$ of a Menger algebra of the rank $n+1$ such that

$$x[e_0, \ldots, e_n] = x, \quad e_i[x_0, \ldots, x_n] = x_i, \quad i = 0, \ldots, n, \tag{5}$$

for all elements $x, x_0, \ldots, x_n$ of the Menger algebra.
They also are called *selectors,* and Menger algebras having a full collection of selectors are called *unitary.* If a unitary Menger algebra has unary operations, then it will be called *extended.*

Consider binary superpositions of $(n+1)$-ary operations $\underset{0}{\oplus}, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus}$, which many authors used in their investigations, in particular Mann (see [9]), V.D. Belousov (see [1]–[3]) and many other authors (some review of such works can be found in [6], [7]). This superpositions are defined by the equalities

$$\left(f \underset{i}{\oplus} h\right)(a_0, \ldots, a_n) := f(a_0, \ldots, a_{i-1}, h(a_0, \ldots, a_n), a_{i+1}, \ldots, a_n), \quad i = 0, \ldots, n. \tag{6}$$

Sometimes they are called Mann superpositions, more precisely $\underset{i}{\oplus}$ is called the *i-th (Mann) superposition.* Each of the operations $\underset{i}{\oplus}$, $i = 0, 1, \ldots, n$, is associative. So, an algebra $(\Phi; \underset{0}{\oplus}, \underset{1}{\oplus}, \ldots, \underset{n}{\oplus})$, where $\Phi \subseteq \Gamma_n(Q)$, will be called *multisemigroup of* $(n+1)$*-ary operations.* When $n = 1$, it is known as a *bisemigroup*, and when $n = 0$, the algebra $(\Gamma_0(Q); \underset{0}{\oplus})$ coincides with the semigroup of all transformations of the set Q.

It is easy to prove that the binary superpositions and selectors are connected by the following equalities:

$$f \underset{i}{\oplus} e_i = f, \qquad e_j \underset{i}{\oplus} g = \begin{cases} g, & \text{if} \quad j = i; \\ e_j, & \text{if} \quad j \neq i. \end{cases} \tag{7}$$

K.A. Zaretsky (in [15]) found an abstract characterization of the class of all bisemigroups consisting of all binary operations of a set. T. Yakubov (in [14]) generalized the mentioned result for the n-ary case. The main idea of their method is in using all constant operations.

V.S. Trokhimenko (in [13]) found an abstract characterizations of the class of all Menger bisemigroups and Menger semigroups of binary operations and some of their subclasses. He also emphasized that the axiomatic finding problem of the class of all multisemigroups was still open. The other problem, stated in [13], deals with the finding of main abstract relations between the commutation of operations and different kinds of superpositions.

The other method for solving these problems is proposed in [12]. It permits to find abstract characterizations of classes of multisemigroups of operations; of Menger multisemigroups of operations and some of their special subclasses.

Here we suggest the improved method and solve the other Trokhimenko's problems. Namely, we give main abstract properties of commutation of operations and their relationships with Menger and Mann superpositions. As a corollary we obtained the results from the article [12].

Note, that we follow the traditions in named of algebras. Namely, the algebras of an abstract class of operations are called by the same way but without the word "operations". For example, "a Menger multisemigroup of $(n+1)$-ary operations" means an algebra, which consists of $(n+1)$-ary operations defined on a set; and "Menger multisemigroup" means an algebra belonging to the abstract class of all Menger multisemigroups of $(n+1)$-ary operations, i.e. the algebra is isomorphic to a Menger multisemigroup of operations defined on a set.

Everywhere below the expression a_i^j denotes the sequence $a_i, \ldots, a_j$, when $i \leqslant j$, and the empty sequence, when $i > j$.

1. Unars of commutations

An abstract characterization of the class of all unars of operation commutations is given in the following fact.

Theorem 1. *A unar is isomorphic to a unar of commutations of $(n+1)$-ary operations if and only if its signature generates a group embedding in a symmetrical group of the degree $n+1$.*

Proof. One part of the theorem follows from (1) immediately.

Conversely, let a unar $(G; V)$ satisfy the conditions of the theorem, i.e. the group $\langle V\rangle$ generated by the set V of transformations of G is isomorphic to a subgroup U of the symmetrical group of the set $\{0, 1, \ldots, n\}$, and let φ denotes an isomorphism from $\langle V\rangle$ onto U and ε denotes the identical transformation. Since the groups $\langle V\rangle$ and U are isomorphic, $\langle V\rangle$ is a finite group of substitutions of the set G, and therefore $\langle V\rangle$ determines an action on G, i.e. the relationships

$$\varepsilon g = g, \qquad \sigma(\tau g) = (\sigma\tau)g \tag{8}$$

hold for all elements $g \in G$ and $\sigma, \tau \in \langle V\rangle$.

Let $e_0, \ldots, e_n, c$ be pairwise different elements not belonging to the set G and let

$$G_0 := G \cup \{e_0, \ldots, e_n, c\}.$$

To every element $g \in G$ we assign an $(n+1)$-ary operation $P(g)$ determined on the set G_0:

$$P(g)(x_0, \ldots, x_n) := \begin{cases} \sigma g, & \text{if } x_0 = e_{\varphi(\sigma)0}, \ldots, x_n = e_{\varphi(\sigma)n}; \\ c, & \text{otherwise.} \end{cases} \tag{9}$$

At first, we prove that the mapping P is an isomorphism between the algebras $(G; \langle V\rangle)$ and $(P(G); U)$.

If $P(g_1) = P(g_2)$, then this in particular means that

$$P(g_1)(e_0, \ldots, e_n) = P(g_2)(e_0, \ldots, e_n),$$

i.e.

$$P(g_1)\left(e_{\varphi(\varepsilon)0}, \ldots, e_{\varphi(\varepsilon)n}\right) = P(g_2)\left(e_{\varphi(\varepsilon)0}, \ldots, e_{\varphi(\varepsilon)n}\right).$$

According to (9), we obtain $\varepsilon g_1 = \varepsilon g_2$, therefore $g_1 = g_2$. Hence, the mapping P is injective. It is surjective too, because $P(G)$ is the set of all images under the mapping P.

Now we will prove the homomorphism property of P, i.e. we will examine the equality

$$P(\tau g) = \varphi(\tau) P(g), \tag{10}$$

for all $\tau \in \langle V\rangle$. Indeed,

$$\begin{aligned}(\varphi(\tau)P(g))\left(e_{\varphi(\sigma)0}, \ldots, e_{\varphi(\sigma)n}\right) &\overset{(1)}{=} P(g)\left(e_{\varphi(\sigma)\varphi(\tau)0}, \ldots, e_{\varphi(\sigma)\varphi(\tau)n}\right) = \\ &= P(g)\left(e_{\varphi(\sigma\tau)0}, \ldots, e_{\varphi(\sigma\tau)n}\right) \overset{(9)}{=} (\sigma\tau)g = \\ &\overset{(8)}{=} \sigma(\tau g) \overset{(9)}{=} P(\tau g)\left(e_{\varphi(\sigma)0}, \ldots, e_{\varphi(\sigma)n}\right).\end{aligned}$$

At last, let a tuple $(x_0,\dots,x_n)$ does not fall under the considered cases, so the tuple $(x_{\varphi(\tau)0},\dots,x_{\varphi(\tau)n})$ has the same property, therefore

$$(\varphi(\tau)P(g))(x_0,\dots,x_n) \overset{(1)}{=} P(g)(x_{\varphi(\tau)0},\dots,x_{\varphi(\tau)n}) \overset{(9)}{=} c \overset{(9)}{=} P(\tau g)(x_0,\dots,x_n).$$

The established isomorphism between the algebras $(G;\langle V\rangle)$ and $(P(G);U)$ determines an isomorphism between the algebras $(G;V)$ and $(P(G);\varphi(V))$, since $\varphi(V)\subset U$. ■

Note. Theorem 1 implies that the assigning of the corresponding commutation of operations to a permutation of $\{0,\ 1,\ \dots,\ n\}$ is an one-valued mapping and it is an abstract property. An image of this mapping will be denoted by the same symbol. This will not lead to misunderstanding, since the unary operations and the corresponding permutations are determined on different sets.

2. Menger superpositions and commutations

We find abstract connections between Menger superpositions and the commutations.

Theorem 2. *An algebra $(G;\mathcal{O},V)$ is isomorphic to an extended Menger algebra of $(n+1)$-ary operations if and only if the operation $\mathcal{O}$ is superassociative, the conditions of Theorem 1 are true in $(G;V)$ and the identities*

$$(\sigma x)[x_0,\dots,x_n] = x[x_{\sigma 0},\dots,x_{\sigma n}], \tag{11}$$

and

$$\sigma(x[x_0,\dots,x_n]) = x[\sigma x_0,\dots,\sigma x_n] \tag{12}$$

are valid for any $\sigma\in V$ and for all x, $x_0,\dots,\ x_n\in G$.

Proof. Let $(\Phi;\mathcal{O},U)$ be an arbitrary extended Menger algebra of $(n+1)$-ary operations of a set Q. Suppose that $f,\ f_0,\ \dots,\ f_n$ are operations from the set Φ and $\sigma\in U$. Then for any elements $a_0,\ \dots,\ a_n\in Q$ we have:

$$\begin{aligned}(\sigma f)[f_0,\dots,f_n](a_0^n) &\overset{(2)}{=} (\sigma f)\big(f_0(a_0^n),\dots,f_n(a_0^n)\big) = \\ &\overset{(1)}{=} f\big(f_{\sigma 0}(a_0^n),\dots,f_{\sigma n}(a_0^n)\big) = \\ &\overset{(2)}{=} f[f_{\sigma 0},\dots,f_{\sigma n}](a_0^n).\end{aligned}$$

It means the validity of the identity (11) in $(\Phi;\mathcal{O},U)$. Next,

$$\sigma(f[f_0,\dots,f_n])(a_0^n) \overset{(1)}{=} f[f_0,\dots,f_n](a_{\sigma 0},\dots,a_{\sigma n}) =$$

$$\overset{(2)}{=} f\big(f_0(a_{\sigma 0},\dots,a_{\sigma n}),\dots,f_n(a_{\sigma 0},\dots,a_{\sigma n})\big) =$$

$$\overset{(1)}{=} f\big(\sigma f_0(a_0^n),\dots,\sigma f_n(a_0^n)\big) \overset{(2)}{=} f[\sigma f_0,\dots,\sigma f_n](a_0^n).$$

This is a proof of the identity (12). The truth of superassociativity is a defining property of a Menger algebra of operations.

Vice versa, let an algebra $(G;\mathcal{O},V)$ satisfy the conditions of the theorem. Henceforth, we will follow the notations of Theorem 1.

For every $g \in G$ we determine an $(n+1)$-ary operation $P(g)$ on the set G_0:

$$P(g)(x_0,\dots,x_n) := \begin{cases} g[x_0,\dots,x_n], & \text{if } x_0,\dots,x_n \in G; \quad \text{(i)} \\ \sigma g, & \text{if } x_0 = e_{\sigma 0},\dots,x_n = e_{\sigma n}; \quad \text{(ii)} \\ c, & \text{otherwise.} \quad \text{(iii)} \end{cases}$$

Now we will prove the equality (10). The proofs of the cases (ii) and (iii) repeat the corresponding reasoning from the proof of Theorem 1, therefore we shall only consider the case (i):

$$\begin{aligned}\tau P(g)(x_0,\dots,x_n) &\overset{(1)}{=} P(g)(x_{\tau 0},\dots,x_{\tau n}) =\\ &\overset{(\mathrm{i})}{=} g[x_{\tau 0},\dots,x_{\tau n}] \overset{(11)}{=} (\tau g)[x_0,\dots,x_n] =\\ &\overset{(\mathrm{i})}{=} P(\tau g)(x_0,\dots,x_n).\end{aligned}$$

Hence, (10) holds for an arbitrary $\tau \in \langle V\rangle$.

To prove that the mapping P has the homomorphism property with respect to the operation $\mathcal{O}$, we have to consider the correctness of the equation

$$P(g[g_0,\dots,g_n]) = P(g)[P(g_0),\dots,P(g_n)] \tag{13}$$

for arbitrary $g, g_0, ...g_n \in G$.

According to the definition of $P(g)$, we have to consider the three cases.

(α) Let x_0, …, x_n be arbitrary elements from the set G, then:

$$\begin{aligned}P(g[g_0,\dots,g_n])(x_0,\dots,x_n) &\overset{(\mathrm{i})}{=} \big(g[g_0,\dots,g_n]\big)[x_0,\dots,x_n] =\\ &\overset{(3)}{=} g\big[g_0[x_0,\dots,x_n],\dots,g_n[x_0,\dots,x_n]\big] =\\ &\overset{(\mathrm{i})}{=} P(g)\big(g_0[x_0,\dots,x_n],\dots,g_n[x_0,\dots,x_n]\big) =\\ &\overset{(\mathrm{i})}{=} P(g)\big(P(g_0)(x_0^n),\dots,P(g_n)(x_0^n)\big) =\\ &\overset{(2)}{=} P(g)\big[P(g_0),\dots,P(g_n)\big](x_0^n).\end{aligned}$$

(β) Let $x_0 = e_{\tau 0}$, ..., $x_n = e_{\tau n}$. Then

$$\begin{aligned}
&P(g[g_0,\ldots,g_n])\,(e_{\tau 0},\ldots,e_{\tau n}) \overset{\text{(ii)}}{=} \tau(g[g_0,\ldots,g_n]) = \\
&\overset{(12)}{=} g[\tau g_0,\ldots,\tau g_n] \overset{\text{(i)}}{=} P(g)(\tau g_0,\ldots,\tau g_n) = \\
&\overset{\text{(ii)}}{=} P(g)\,(P(g_0)\,(e_{\tau 0},\ldots,e_{\tau n})\,,\ldots,P(g_n)\,(e_{\tau 0},\ldots,e_{\tau n})) = \\
&\overset{(2)}{=} P(g)[P(g_0),\ldots,P(g_n)]\,(e_{\tau 0},\ldots,e_{\tau n})\,.
\end{aligned}$$

(γ) Otherwise, we obtain

$$\begin{aligned}
&P(g[g_0,\ldots,g_n])(x_0^n) \overset{\text{(iii)}}{=} c \overset{\text{(iii)}}{=} P(g)(c,\ldots,c) = \\
&\overset{\text{(iii)}}{=} P(g)\big(P(g_0)(x_0^n),\ldots,P(g_n)(x_0^n)\big) = \\
&\overset{(2)}{=} P(g)\big[P(g_0),\ldots,P(g_n)\big](x_0^n).
\end{aligned}$$

Hence, the equality

$$P(g[g_0,\ldots,g_n])(x_0,\ldots,x_n) = P(g)\big[P(g_0),\ldots,P(g_n)\big](x_0,\ldots,x_n)$$

holds for all elements x_0, ..., x_n from G_0. It means, that equation (13) is true.

Therefore, the mapping P is a homomorphism from the algebra $(G;\mathcal{O},\langle V\rangle)$ onto the constructed operation algebra $(P(G);\mathcal{O},U)$. The equality $P(g_0) = P(g_1)$ implies

$$P(g_0)\,(e_0,\ldots,e_n) = P(g_1)\,(e_0,\ldots,e_n)\,,$$

i.e. $g_0 = g_1$. Therefore, the mapping P is injective. The surjectivity follows from the definition of the set $P(G)$.

Thus, the mapping P determines an isomorphism between the algebras $(G;\mathcal{O},\langle V\rangle)$ and $(P(G);\mathcal{O},U)$. It is easy to see that it is an isomorphism between the algebras $(G;\mathcal{O},V)$ and $(P(G);\mathcal{O},\varphi(V))$ too, where φ denotes an isomorphism of the groups $\langle V\rangle$ and U (see the proof of Theorem 1). ∎

By setting $V=\{\varepsilon\}$ in this theorem, we obtain the correctness of the following well known statement.

Theorem 3 (see [11]). *An $(n+1)$-ary groupoid is isomorphic to a Menger algebra of n-ary operations rank n if and only if it is superassociative.*

An abstract characterization of the class of extended unitary Menger algebras of operations is given as follows:

Theorem 4. *A universal algebra is isomorphic to an extended unitary Menger algebra of $(n+1)$-ary operations if and only if the conditions of Theorem* 2 *and the identities* (5) *hold.*

Proof. Standard verification. ∎

Corollary 5 (see [11]). *A universal algebra is isomorphic to a unitary Menger algebra of operations if and only if it is a unitary Menger algebra.*

3. Binary algebras

Let $(G;\underset{0}{\circ},\dots,\underset{n}{\circ})$ be an arbitrary binary algebra, i.e. an algebra, whose signature consists of binary operations only. Let us introduce the following agreements:

I. To short writings we put

$$(14)\quad x\overset{i_s}{\underset{i_0}{\circ}}y_0^s := x\underset{i_0}{\circ}y_0\underset{i_1}{\circ}y_1\underset{i_2}{\circ}\dots\underset{i_s}{\circ}y_s := \left(\dots\left(\left(x\underset{i_0}{\circ}y_0\right)\underset{i_1}{\circ}y_1\right)\underset{i_2}{\circ}\dots\right)\underset{i_s}{\circ}y_s.$$

II. Consider an algebra $(G;\underset{0}{\circ},\dots,\underset{n}{\circ},V)$, where $(G;\underset{0}{\circ},\dots,\underset{n}{\circ})$ is a binary algebra and V is a set of unary operations defined on G. We define partial operations $\underset{0}{\bullet},\ \dots,\ \underset{n}{\bullet}$ on the set $G_0:=G\cup\{e_0,\dots,e_n,c\}, e_0,\dots,e_n,c\neq G$, by the condition

$$(15)\qquad x \underset{i}{\bullet} y := \begin{cases} x \underset{i}{\circ} y, & \text{if} \quad x, y \in G; \\ y, & \text{if} \quad x = e_i; \\ e_j, & \text{if} \quad x = e_j \text{ and } j \neq i; \\ x, & \text{if} \quad y = e_i; \\ c, & \text{otherwise.} \end{cases}$$

The elements $e_0, \dots, e_n$ are called *selectors*. Our aim does not need to define $\sigma(x)$ for $x \notin G$. It is easy to verify the property

$$(16)\qquad e_i \overset{i_s}{\underset{i_0}{\bullet}} y_0^s = \begin{cases} y_p \underset{i_{p+1}}{\bullet} y_{p+1} \underset{i_{p+2}}{\bullet} \dots \underset{i_s}{\bullet} y_s, & \text{if } i_p = i, \\ & \text{but } i_k \neq i \text{ for all } k < p; \\ e_i, & \text{if } i \notin \{i_0, \dots, i_s\} \end{cases}$$

in the algebra $(G_0; \underset{0}{\bullet}, \dots, \underset{n}{\bullet}, V)$. The algebra $(\Phi; \underset{0}{\oplus}, \underset{1}{\oplus}, \dots, \underset{n}{\oplus}, U)$ of operations, where (Φ, U) is above and $\underset{i}{\oplus}$ are Mann superpositions, is an example of such an algebra. The selectors are defined by the equalities (1).
Let $(G_0; \underset{0}{\bullet}, \dots, \underset{n}{\bullet}, V)$ be an algebra as above such that the unary operations of V are invertible.

Then, a tuple $(x_0, \dots, x_n)$ of elements of this algebra is said to be (σ, i)*-labelled*, $i \in \{0, 1, \dots, n\}, \sigma \in V$, if there exist elements $y, y_0, \dots, y_s$ and operations $\underset{i_0}{\bullet}, \dots, \underset{i_s}{\bullet}$ such that

$$(17)\qquad x_{\sigma^{-1}(j)} = \left(e_j \underset{i}{\bullet} y\right) \overset{i_s}{\underset{i_0}{\bullet}} y_0^s$$

for all $j = 0, \dots, n$. In this case the element $x_{\sigma^{-1}(i)}$ is called *labelled*.

A tuple $(x_0, \ldots, x_n)$ is called labelled, if there are σ and i as above such that $(x_0, \ldots, x_n)$ is (σ, i)-labelled.

Note that the set of all labelled tuples is not empty. For example, every tuple of the type $(e_0^{i-1}, g, e_{i+1}^n)$, where $g \in G$, is (ε, i)-labelled, since $e_j \underset{i}{\bullet} g = e_j$, for all $j \neq i$. Definition (15) implies that the tuple $(e_0, \ldots, e_n)$ is (ε, i)-labelled for every $i = 0, \ldots, n$.

We also need the following statement.

Proposition 6. *For arbitrary elements of an algebra* $(G_0; \underset{0}{\bullet}, \ldots, \underset{n}{\bullet}, V)$ *the next statements are true:*

(a) *if a tuple* $(x_0, \ldots, x_n)$ *is* (σ, i)*-labelled* (see (17)), *then the tuple* $(x_{\tau(0)}, \ldots, x_{\tau(n)})$ *is* $(\sigma\tau, i)$*-labelled;*

(b) *if a tuple* $(x_0, \ldots, x_n)$ *is not labelled, then for arbitrary* τ *the tuple* $(x_{\tau 0}, \ldots, x_{\tau n})$ *will not be labelled too;*

(c) *if a tuple* $(x_0, \ldots, x_n)$ *is* (σ, i)*-labelled* (see (17)), *then the tuple*

$$\left(x_0, \ldots, x_{\sigma^{-1}(m)-1}, \left(g \underset{i}{\bullet} y\right) \overset{i_s}{\underset{i_0}{\bullet}} y_0^s, x_{\sigma^{-1}(m)+1}, \ldots, x_n\right)$$

is (σ, m)*-labelled, for any* $m = 0, 1, \ldots, n$.

Proof. (a): Rename the elements in the tuple $(x_{\tau 0}, \ldots, x_{\tau n})$ according to the order of ascending its indices:

$$(y_0, \ldots, y_n) = (x_{\tau 0}, \ldots, x_{\tau n}),$$

i.e. put $y_k := x_{\tau k}$ for all $k = 0, \ldots, n$. Then $y_{\tau^{-1}(k)} = x_k$. Set here $k := \sigma^{-1}(j)$, hence

$$y_{\tau^{-1}\sigma^{-1}(j)} = x_{\sigma^{-1}(j)}, \qquad j = 0, 1, \ldots, n.$$

Comparing with (17), we obtain

$$y_{(\sigma\tau)^{-1}(j)} = \left(e_j \underset{i}{\bullet} y\right) \overset{i_s}{\underset{i_0}{\bullet}} y_0^s$$

for all $j = 0, \ldots, n$, i.e. the tuple $(y_0, \ldots, y_n)$ is $(\sigma\tau, i)$-labelled.

(b): Suppose that a tuple $(x_{\tau 0}, \ldots, x_{\tau n})$ is (σ, i)-labelled for some parameters σ and i, then, by (a), the tuple $(x_0, \ldots, x_n)$ is on $(\sigma\tau^{-1}, i)$-labelled. This is a contradiction to the assumption.

(c): If $j \neq m$, then

$$\left(\left(e_j \underset{m}{\bullet} g\right) \underset{i}{\bullet} y\right) \overset{i_s}{\underset{i_0}{\bullet}} y_0^s \overset{(15)}{=} \left(e_j \underset{i}{\bullet} y\right) \overset{i_s}{\underset{i_0}{\bullet}} y_0^s \overset{(17)}{=} x_{\sigma^{-1}(j)}.$$

If $j = m$, then

$$\left(\left(e_j \underset{m}{\bullet} g\right) \underset{i}{\bullet} y\right) \overset{i_s}{\underset{i_0}{\bullet}} y_0^s \overset{(15)}{=} \left(g \underset{i}{\bullet} y\right) \overset{i_s}{\underset{i_0}{\bullet}} y_0^s \overset{(17)}{=} x_{\sigma^{-1}(j)}.$$

Hence, we have obtained an element, which is on the $\sigma^{-1}(m)$-th place in the tuple $(x_0, \ldots, x_n)$. ∎

To find an abstract characterization of the class of extended multisemigroups of operations we need the following

Lemma 7. *For arbitrary operations f, h_0, h_1, $\ldots$, $h_s \in \Gamma(Q)$, for an arbitrary sequence of nonnegative integers i_0, $\ldots$, i_s and for arbitrary elements a_0, $\ldots$, $a_n \in Q$ the equation*

$$\left(f \overset{i_s}{\underset{i_0}{\oplus}} h_0^s\right)(a_0^n) = f\left(e_0 \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n), \ldots, e_n \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n)\right) \tag{18}$$

holds (see (14) for notations).

Proof. We prove the statement by induction in s. When $s = 0$, we have

$$\left(f \underset{i}{\oplus} h\right)(a_0, \ldots, a_n) \overset{(6)}{=} f(a_0, \ldots, a_{i-1}, h(a_0, \ldots, a_n), a_{i+1}, \ldots, a_n) =$$

$$\overset{(4)}{=} f(e_0(a_0^n), \ldots, e_{i-1}(a_0^n), h(a_0^n), e_{i+1}(a_0^n), \ldots, e_n(a_0^n)) =$$

$$\overset{(7)}{=} f\left(\left(e_0 \underset{i}{\oplus} h\right)(a_0^n), \ldots, \left(e_i \underset{i}{\oplus} h\right)(a_0^n), \ldots, \left(e_n \underset{i}{\oplus} h\right)(a_0^n)\right).$$

Suppose the lemma is true for s, i.e. (18) holds for all operations, number sequences and elements of the set Q occurring in the lemma.

Let ξ be an arbitrary operation from $\Gamma(Q)$ and let $i \in \{0, \dots, n\}$. Then, by the inductive assumption, we obtain

$$\left(\left(\xi \underset{i}{\oplus} f\right) \overset{i_s}{\underset{i_0}{\oplus}} h_0^s\right)(a_0, \dots, a_n) \overset{(18)}{=} \left(\xi \underset{i}{\oplus} f\right)\left(e_0 \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n), \dots, e_n \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n)\right) =$$

$$\overset{(6)}{=} \xi\left(e_0 \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n), \dots, e_{i-1} \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n), f\left(e_0 \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n), \dots, e_n \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n)\right),\right.$$

$$\left. e_{i+1} \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n), \dots, e_n \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n)\right) =$$

$$\overset{(18)}{=} \xi\left(e_0 \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n), \dots, e_{i-1} \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n), f \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n), e_{i+1} \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n),\right.$$

$$\left. \dots, e_n \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n)\right) =$$

$$\overset{(7)}{=} \xi\left(\left(e_0 \underset{i}{\oplus} f\right) \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n), \dots, \left(e_{i-1} \underset{i}{\oplus} f\right) \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n), \left(e_i \underset{i}{\oplus} f\right) \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n),\right.$$

$$\left. \left(e_{i+1} \underset{i}{\oplus} f\right) \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n), \dots, \left(e_n \underset{i}{\oplus} f\right) \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n)\right).$$

Hence, the equality

$$\left(\left(\xi \underset{i}{\oplus} f\right) \overset{i_s}{\underset{i_0}{\oplus}} h_0^s\right)(a_0^n) = \xi\left(\left(e_0 \underset{i}{\oplus} f \overset{i_s}{\underset{i_0}{\oplus}} h_0^s\right)(a_0^n), \dots, \left(e_n \underset{i}{\oplus} f \overset{i_s}{\underset{i_0}{\oplus}} h_0^s\right)(a_0^n)\right)$$

holds for all elements $a_0, \dots, a_n \in Q$. Thus, the lemma's statement is true for $s + 1$. Therefore, according to the mathematical induction theorem, the equality (18) is valid for every nonnegative integer s. ■

Let Φ be a set of $(n+1)$-ary operations defined on an arbitrary fixed set Q and let Φ be closed under the Mann superpositions $\underset{0}{\oplus}, \dots, \underset{n}{\oplus}$ and under every commutation of a set $U \subseteq S_{n+1}$. Then the algebra $(\Phi; \underset{0}{\oplus}, \ldots, \underset{n}{\oplus}, U)$ will be called *extended multisemigroup of* $(n+1)$*-ary operations*. An abstract characterization of the class of all extended multisemigroups of operations is given in the following ststment.

Theorem 8. *An algebra* $(G; \underset{0}{\circ}, \ldots, \underset{n}{\circ}, V)$ *is isomorphic to an extended multisemigroup of* $(n+1)$*-ary operations if and only if it satisfies the conditions of Theorem* 1 *as well as the following relations are true:*

$$\sigma\left(x \underset{i}{\circ} y\right) = \sigma x \underset{\sigma i}{\circ} \sigma y \tag{19}$$

and

$$\left(\bigwedge_{i=0}^{i=n} e_{\sigma i} \overset{i_s}{\underset{i_0}{\bullet}} y_0^s = e_{\tau i} \overset{j_m}{\underset{j_0}{\bullet}} z_0^m\right) \Longrightarrow (\sigma x) \overset{i_s}{\underset{i_0}{\circ}} y_0^s = (\tau x) \overset{j_m}{\underset{j_0}{\circ}} z_0^m \tag{20}$$

for all x, y_0, $\ldots$, y_s, z_0, $\ldots$, $z_m \in Q$ *and for all* τ, $\sigma \in V$ (see (16)).

Note. In (19) and (20) σx denotes the image of x under $\sigma \in V$, but σi denotes the image of the number $i \in \{0, \ldots, n\}$ under the substitution σ of the set $\{0, \ldots, n\}$, which corresponds to $\sigma \in V$ according to Theorem 1.

Proof. Let $(\Phi; \underset{0}{\oplus}, \ldots, \underset{n}{\oplus}, U)$ be an arbitrary extended multisemigroup of $(n+1)$-ary operations defined on a set Q. The truth of (19) is proved by the following equalities:

$$\sigma\left(f \underset{i}{\oplus} h\right)(a_0, \ldots, a_n) \overset{(1)}{=} \left(f \underset{i}{\oplus} h\right)(a_{\sigma 0}, \ldots, a_{\sigma n}) =$$

$$\overset{(6)}{=} f(a_{\sigma 0}, \ldots, a_{\sigma(i-1)}, h(a_{\sigma 0}, \ldots, a_{\sigma n}), a_{\sigma(i+1)}, \ldots, a_{\sigma n}) =$$

$$\overset{(1)}{=} (\sigma f)(a_0, \ldots, a_{\sigma(i)-1}, \sigma h(a_0, \ldots, a_n), a_{\sigma(i)+1}, \ldots, a_n) =$$

$$\overset{(6)}{=} \left(\sigma f \underset{\sigma i}{\oplus} \sigma h\right)(a_0, \ldots, a_n).$$

To prove the relation (20), we define operations $(\underset{0}{\bullet}),\ldots,(\underset{n}{\bullet})$ on the set Φ_0 by (15), where $(\underset{0}{\circ}) = (\underset{0}{\oplus}),\ldots,(\underset{n}{\circ}) = (\underset{n}{\oplus})$, and $e_0,\ldots, e_n$ are trivial operations defined by (4) and $c \notin \Phi$ is an arbitrary symbol. We assume that

$$\bigwedge_{i=0}^{i=n} e_{\sigma i} \overset{i_s}{\underset{i_0}{\bullet}} h_0^s = e_{\tau i} \overset{j_m}{\underset{j_0}{\bullet}} \chi_0^m \tag{21}$$

for some operations $h_i, \chi_i \in \Phi$ and permutations σ, $\tau \in U$. Since $h_i, \chi_i \in \Phi$ and (7),

$$e_{\sigma i} \overset{i_s}{\underset{i_0}{\bullet}} h_0^s = e_{\sigma i} \overset{i_s}{\underset{i_0}{\oplus}} h_0^s, \qquad e_{\tau i} \overset{j_m}{\underset{j_0}{\bullet}} \chi_0^m = e_{\tau i} \overset{j_m}{\underset{j_0}{\oplus}} \chi_0^m$$

for all i=0, 1,…, n. So,

$$\bigwedge_{i=0}^{i=n} e_{\sigma i} \overset{i_s}{\underset{i_0}{\oplus}} h_0^s = e_{\tau i} \overset{j_m}{\underset{j_0}{\oplus}} \chi_0^m \tag{22}$$

and for every $(n+1)$-ary operation f we obtain

$$\begin{aligned}
\left(\sigma f \overset{i_s}{\underset{i_0}{\oplus}} h_0^s\right)(a_0^n) &\overset{(18)}{=} \sigma f\left(e_0 \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n),\ldots, e_n \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n)\right) = \\
&\overset{(1)}{=} f\left(e_{\sigma 0} \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n),\ldots, e_{\sigma n} \overset{i_s}{\underset{i_0}{\oplus}} h_0^s(a_0^n)\right) = \\
&\overset{(22)}{=} f\left(e_{\tau 0} \overset{j_m}{\underset{j_0}{\oplus}} \chi_0^m(a_0^n),\ldots, e_{\tau n} \overset{j_m}{\underset{j_0}{\oplus}} \chi_0^m(a_0^n)\right) = \\
&\overset{(1)}{=} \tau f\left(e_0 \overset{j_m}{\underset{j_0}{\oplus}} \chi_0^m(a_0^n),\ldots, e_n \overset{j_m}{\underset{j_0}{\oplus}} \chi_0^m(a_0^n)\right) = \\
&\overset{(18)}{=} \left(\tau f \overset{j_m}{\underset{j_0}{\oplus}} \chi_0^m\right)(a_0^n).
\end{aligned}$$

Since the elements $a_0, \ldots, a_n$ are arbitrary, one obtains

$$\sigma f \overset{i_s}{\underset{i_0}{\oplus}} h_0^s = \tau f \overset{j_m}{\underset{j_0}{\oplus}} \chi_0^m .$$

So, the implication (20) holds in any multisemigroup of operations.

Vice versa, let an algebra $(G; \underset{0}{\circ}, \ldots, \underset{n}{\circ}, V)$ satisfy the conditions of the theorem. Henceforth, we will follow the notations of the proof of Theorem 1.

To every element $g \in G$ we will assign an $(n+1)$-ary operation $P(g)$ determined on the set G_0:

$$P(g)(x_0^n) :=$$

(23)

$$= \begin{cases} (\sigma g) \underset{j}{\bullet} y \overset{i_s}{\underset{i_0}{\bullet}} y_0^s, & \text{if the tuple is } (\sigma, j)\text{-labelled (see (17))}; \quad \text{(i)} \\ c, & \text{otherwise.} \quad \text{(ii)} \end{cases}$$

To prove that $P(g)$ is an operation for every $g \in G$, we have to establish that for every tuple $(x_0, \ldots, x_n)$ the result $P(g)(x_0, \ldots, x_n)$ is uniquely determined, i.e. it depends neither on the choice of a labelled element, nor on its decomposition, nor on a permutation σ.

Let a tuple $(x_0, \ldots, x_n)$ be (σ, i)- and (τ, j)-labelled, that is

$$x_{\sigma^{-1}(k)} = e_k \underset{i}{\bullet} y \overset{i_s}{\underset{i_0}{\bullet}} y_0^s, \qquad x_{\tau^{-1}(k)} = e_k \underset{j}{\bullet} z \overset{j_m}{\underset{j_0}{\bullet}} z_0^m$$

for all $k = 0, \ldots, n$. Replacing the indices, we obtain

$$\bigwedge_{k=0}^{n} e_{\sigma k} \underset{i}{\bullet} y \overset{i_s}{\underset{i_0}{\bullet}} y_0^s = x_k = e_{\tau k} \underset{j}{\bullet} z \overset{j_m}{\underset{j_0}{\bullet}} z_0^m .$$

It means the truth of the antecedent of the implication (20), therefore the consequent of (20) holds, i.e.

$$(\sigma g) \underset{i}{\bullet} y \overset{i_s}{\underset{i_0}{\bullet}} y_0^s = (\tau g) \underset{j}{\bullet} z \overset{j_m}{\underset{j_0}{\bullet}} z_0^m .$$

Thus $P(g)$ is an operation for an arbitrary element $g \in G$.

Now let us prove that P has the homomorphism property. According to (23), for proving both of the equalities (10) and

$$P\left(g \underset{i}{\circ} h\right) = P(g) \underset{i}{\oplus} P(h), \qquad i = 0, 1, \ldots, n \tag{24}$$

two cases (i) and (ii) have to be considered.

(i): Let $(x_0, \ldots, x_n)$ be (σ, j)-labelled (see (17)), then

$$P(\tau g)(x_0, \ldots, x_n) \overset{\text{(i)}}{=} \sigma(\tau g) \underset{j}{\bullet} y \overset{i_s}{\underset{i_0}{\bullet}} y_0^s \overset{(8)}{=} (\sigma\tau g) \underset{j}{\bullet} y \overset{i_s}{\underset{i_0}{\bullet}} y_0^s.$$

Proposition 6 implies that the tuple $(x_{\tau 0}, \ldots, x_{\tau n})$ is $(\sigma\tau, i)$-labelled, therefore

$$\tau P(g)(x_0, \ldots, x_n) \overset{(1)}{=} P(g)(x_{\tau 0}, \ldots, x_{\tau n}) \overset{\text{(i)}}{=} (\sigma\tau g) \underset{j}{\bullet} y \overset{i_s}{\underset{i_0}{\bullet}} y_0^s.$$

The right parts of the equalities are equal, so it implies the equality of the left ones. Therefore, in the case (i), property (10) has been proved.

(ii): If the tuple $(x_0, \ldots, x_n)$ is not labelled, then, by Proposition 6, the tuple $(x_{\tau 0}, \ldots, x_{\tau n})$ is not labelled too. Therefore,

$$\tau P(g)(x_0, \ldots, x_n) \overset{(1)}{=} P(g)(x_{\tau 0}, \ldots, x_{\tau n}) \overset{\text{(ii)}}{=} c \overset{\text{(ii)}}{=} P(\tau g)(x_0, \ldots, x_n)$$

and equality (10) holds.

Let us now consider the equality (24). Let $(x_0, \ldots, x_n)$ be (σ, j)-labelled, then proving the first case of (23) we have:

$$\left(P(g) \underset{i}{\oplus} P(h)\right)(x_0^n) \overset{(6)}{=} P(g)\left(x_0^{i-1}, P(h)(x_0, \ldots, x_n), x_{i+1}^n\right) =$$

$$\overset{\text{(i)}}{=} P(g)\left(x_0, \ldots, x_{i-1}, \sigma h \underset{j}{\bullet} y \overset{i_s}{\underset{i_0}{\bullet}} y_0^s, x_{i+1}, \ldots, x_n\right).$$

According to statment (c) of Proposition 6, the tuple

$$(x_0, \dots, x_{i-1}, \sigma h \underset{j}{\bullet} y \overset{i_s}{\underset{i_0}{\bullet}} y_0^s, x_{i+1}, \dots, x_n)$$

is $(\sigma, \sigma i)$-labelled, therefore

$$\left(P(g) \underset{i}{\oplus} P(h)\right)(x_0^n) \overset{\text{(i)}}{=} \left((\sigma g) \underset{\sigma(i)}{\bullet} (\sigma h)\right) \underset{j}{\bullet} y \overset{i_s}{\underset{i_0}{\bullet}} y_0^s =$$

$$\overset{(15)}{=} \left((\sigma g) \underset{\sigma(i)}{\circ} (\sigma h)\right) \underset{j}{\bullet} y \overset{i_s}{\underset{i_0}{\bullet}} y_0^s \overset{(19)}{=} \sigma \left(g \underset{i}{\circ} h\right) \underset{j}{\bullet} y \overset{i_s}{\underset{i_0}{\bullet}} y_0^s =$$

$$\overset{\text{(i)}}{=} P\left(g \underset{i}{\circ} h\right)(x_0, \dots, x_n).$$

If for all $\sigma \in \langle V \rangle$ the tuple $(x_0, \dots, x_n)$ is not labelled, then, by the case (ii) of (23), we have

$$\left(P(g) \underset{i}{\oplus} P(h)\right)(x_0, \dots, x_n) \overset{(6)}{=} P(g)(x_0, \dots, x_{i-1}, P(h)(x_0, \dots, x_n), x_{i+1}, \dots, x_n) =$$

$$\overset{\text{(ii)}}{=} P(g)(x_0, \dots, x_{i-1}, c, x_{i+1}, \dots, x_n) =$$

$$\overset{\text{(ii)}}{=} c \overset{\text{(ii)}}{=} P\left(g \underset{i}{\circ} h\right)(x_0, \dots, x_n).$$

So, the equality (24) is true and therefore, P has the homomorphism property.

P is a surjective mapping from G on $P(G)$, since the set $P(G)$ is determined as a set of all images of the elements from G.

Suppose $P(g) = P(h)$. Since the tuple $(e_0, \dots, e_n)$ is $(\varepsilon, 0)$-labelled,

$$P(g)(e_0, \dots, e_n) = P(h)(e_0, \dots, e_n)$$

implies $g \underset{0}{\bullet} e_0 = h \underset{0}{\bullet} e_0$, i.e. $g = h$. So, P is injective too.

Thus, the mapping P is an isomorphism between the algebras $(G;\underset{0}{\circ},\dots,\underset{n}{\circ},\langle V\rangle)$ and $(P(G);\underset{0}{\oplus},\dots,\underset{n}{\oplus},U)$. Therefore, P is an isomorphism between the algebras $(G;\underset{0}{\circ},\dots,\underset{n}{\circ},V)$ and $(P(G);\underset{0}{\oplus},\dots,\underset{n}{\oplus},\varphi(V))$, where φ denotes an isomorphism of the groups $\langle V\rangle$ and U (see the proof of Theorem 1). ∎

When $V=\{\varepsilon\}$, Theorem 8 implies the next statement, which gives an abstract characterization of the class of all multisemigroups of operations.

Corollary 9 (see [12]). *An algebra $(G;\underset{0}{\circ},\dots,\underset{n}{\circ})$ with the binary operations only is isomorphic to a multisemigroup of $(n+1)$-ary operations, if and only if the following condition*

$$\left(\bigwedge_{i=0}^{i=n} e_i \overset{i_s}{\underset{i_0}{\bullet}} y_0^s = e_i \overset{j_m}{\underset{j_0}{\bullet}} z_0^m\right) \Longrightarrow x \overset{i_s}{\underset{i_0}{\circ}} y_0^s = x \overset{j_m}{\underset{j_0}{\circ}} z_0^m \tag{25}$$

holds for all x, y_0, ..., y_s, z_0, ..., $z_m \in G$ (see (16)). ∎

Note. It is easy to see that (25) implies the associativity of every binary operation $\underset{0}{\circ}$, ..., $\underset{n}{\circ}$. Indeed, if $j\neq i$, then

$$\left(e_j \underset{i}{\bullet} y\right)\underset{i}{\bullet} z \overset{(15)}{=} e_j \overset{(15)}{=} e_j \underset{i}{\bullet}\left(y \underset{i}{\bullet} z\right)$$

and

$$\left(e_i \underset{i}{\bullet} y\right)\underset{i}{\bullet} z \overset{(15)}{=} y \underset{i}{\bullet} z \overset{(15)}{=} e_i \underset{i}{\bullet}\left(y \underset{i}{\bullet} z\right).$$

Therefore, by the formulas (25), (15) and (16) we have:

$$\left(x \underset{i}{\circ} y\right)\underset{i}{\circ} z = x \underset{i}{\circ}\left(y \underset{i}{\circ} z\right),$$

for any $x\in G$. It means the associativity of the operation $\underset{i}{\circ}$, for all $i=0$, 1, ..., n. Hence, every operation of the algebra $(G;\underset{0}{\circ},\dots,\underset{n}{\circ})$ is associative.

4. Menger multisemigroups

An algebra $(\Phi; \mathcal{O}, \underset{0}{\oplus}, \ldots, \underset{n}{\oplus})$ is called a *Menger multisemigroup of* $(n+1)$-*ary operations*, if $\Phi \subseteq \Gamma_n(Q)$. An algebra $(\Phi; \mathcal{O}, \underset{0}{\oplus}, \ldots, \underset{n}{\oplus})$ will be called an *extended Menger multisemigroup of* $(n+1)$-*ary operations*, if $(\Phi; \mathcal{O}, \underset{0}{\oplus}, \ldots, \underset{n}{\oplus})$ is a Menger multisemigroup of $(n+1)$-ary operations and V is a collection of commutations.

By using (16) and Theorems 2 and 8, we have an abstract characterization of the class of all extended Menger multisemigroups of operations of the same arity in the following theorem.

Theorem 10. *An algebra* $(G; \mathcal{O}, \underset{0}{\circ}, \ldots, \underset{n}{\circ}, V)$ *is isomorphic to an extended Menger multisemigroup of* $(n+1)$*-ary operations if and only if* $(G; \mathcal{O}, V)$ *is an extended Menger algebra,* $(G; \underset{0}{\circ}, \ldots, \underset{n}{\circ}, V)$ *is an extended multisemigroup and the following equalites are true:*

$$(\sigma g) \overset{i_s}{\underset{i_0}{\circ}} y_0^s = g\left[e_{\sigma 0} \underset{0}{\bullet} g \overset{i_s}{\underset{i_0}{\circ}} y_0^s, \ldots, e_{\sigma n} \underset{n}{\bullet} g \overset{i_s}{\underset{i_0}{\circ}} y_0^s\right], \tag{26}$$

where $\{i_0, \ldots, i_s\} = \{0, \ldots, n\}$.

Proof. The necessity of the theorem follows from Theorems 1 and 8, and Lemma 7.

To prove its sufficiency we assign to every element $g \in G$ an $(n+1)$-ary operation $P(g)$, being determined on the set G_0 (see the proof of the previous theorem) by the equalities

$$P(g)(x_0^n) := \begin{cases} g[x_0, \ldots, x_n], & \text{if } x_0, \ldots, x_n \in G; & \text{(i)} \\ (\sigma g) \underset{j}{\bullet} y \overset{i_s}{\underset{i_0}{\bullet}} y_0^s, & \text{if the tuple is } (\sigma, j)\text{-labelled;} & \text{(ii)} \\ c, & \text{otherwise.} & \text{(iii)} \end{cases}$$

At first we have to establish the correctness of the definition of $P(g)$. The independence of the result from a permutation σ, a choice of a labelled element and a respective decomposition may be proved in the same way as in Theorem 8.

If a tuple $(x_0, \dots, x_n)$ fulfills the conditions (i) and (ii) simultaneously, then $\{i_0, \dots, i_s\}=\{0, 1, \dots, n\}$ and property (26) guarantees the uniqueness of the result.

The homomorphism property of the mapping P can be proved in the same way as in Theorems 2 and 8 as well as its injectivity. ■

The next corollary follows from this theorem when $V = \{\varepsilon\}$.

Corollary 11 (see [12]). *An algebra $(G; \mathcal{O}, \underset{0}{\circ}, \dots, \underset{n}{\circ})$ is isomorphic to a Menger multisemigroup of $(n+1)$-ary operations if and only if $(G; \mathcal{O})$ is a Menger algebra, $(G; \underset{0}{\bullet}, \dots, \underset{n}{\bullet})$ is a multisemigroup and the relation*

$$g \overset{i_s}{\underset{i_0}{\circ}} y_0^s = g \left[e_0 \underset{0}{\bullet} g \overset{i_s}{\underset{i_0}{\circ}} y_0^s, \dots, e_n \underset{n}{\bullet} g \overset{i_s}{\underset{i_0}{\circ}} y_0^s \right] \tag{27}$$

holds, where $\{i_0, \dots, i_s\} = \{0, \dots, n\}$ (see (16), when $V = \{\varepsilon\}$).

5. Problems

Note that, in fact, the axioms (20), (25)–(27) are shorten writings of a countable family of axioms, every of which is determined by a sequence of nonnegative integers, namely by the indices of the binary operations.

A related recent investigation one can find in [4], [5] and [8]. We emphasize some problems of this theory, which are still unsolved.

1 Find an abstract characterization of the class of all extended unitary multisemigroups of operations, i.e. algebras of the type $(\Phi; \underset{0}{\oplus}, \dots, \underset{n}{\oplus}, U, e_0, \dots, e_n)$ (an abstract characterization of the class of all unitary multisemigroups of operations is unknown, too).

2 Find a description of classes of operation algebras with selected sets of special kinds of operations such as quasigroup operations, prequasigroup operations, continuous, linear, commutative operations and others.

3 Find an abstract characterization of the class all ordered multisemigroups of functions (partial operations), i.e. algebraic systems of the type $(\Phi; \underset{0}{\oplus}, \ldots, \underset{n}{\oplus}, \subseteq)$, where $\subseteq$ is an inclusion (or a partial order) of functions.

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