

ORTHORINGS

IVAN CHAJDA

Palacký University, Olomouc
Department of Algebra and Geometry
Tomkova 40, 77900 Olomouc, Czech Republic

e-mail: chajda@inf.upol.cz

AND

HELMUT LÄNGER

Vienna University of Technology
Institute of Discrete Mathematics and Geometry
Research Unit Algebra
Wiedner Hauptstraße 8–10, 1040 Vienna, Austria

e-mail: h.laenger@tuwien.ac.at

Abstract

Certain ring-like structures, so-called orthorings, are introduced which are in a natural one-to-one correspondence with lattices with 0 every principal ideal of which is an ortholattice. This correspondence generalizes the well-known bijection between Boolean rings and Boolean algebras. It turns out that orthorings have nice congruence and ideal properties.

Keywords: ortholattice, generalized ortholattice, sectionally complemented lattice, orthoring, arithmetical variety, weakly regular variety, congruence kernel, ideal term, basis of ideal terms, subtractive term.

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1. Introduction

The well-known natural bijective correspondence between Boolean algebras and Boolean rings is widely used in applications, see [1] for details. This correspondence was generalized in different ways thus giving rise to natural connections between certain lattice structures on the one hand and certain ring-like structures on the other hand. On the lattice-theoretical side the following structures were considered: orthomodular lattices ([8] and [18]), ortholattices ([2]), bounded lattices with an involutory antiautomorphism ([9], [10], [11], [12], [13] and [14]), pseudocomplemented semilattices ([5]) and *MV*-algebras ([6]). The corresponding ring-like structures were called orthomodular Boolean quasirings or orthomodular pseudorings, orthopseudorings, orthopseudosemirings, Boolean quasirings and pseudorings, respectively. (No name was assigned to the ring-like structures induced by pseudocomplemented semilattices.) In [3] the ring-like structures introduced in [2] and [9], respectively, are related to each other. However, each one of the derived ring-like structures considered so far in this context was endowed with a constant 1 which plays a role similar to the unit element in rings. On the other hand, starting with so-called generalized Boolean algebras, one can derive Boolean rings which need not have a unit element (see [1]). A similar approach was used in [7], where so-called generalized orthomodular lattices (introduced by M.F. Janowitz in [17], see also [16]) were considered. Our aim is to investigate ring-like structures (so-called orthorings) which correspond to lattices with 0 such that every principal lattice ideal is an ortholattice. It should be pointed out that though these lattices do not form a variety, the term equivalent orthorings form a variety and hence allow the application of universal algebraic methods and results. Moreover, we are going to show that – in spite of their generality – orthorings have nice properties.

We recall that an *ortholattice* is an algebra $(L; \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and $(x')' = x$, $(x \vee y)' = x' \wedge y'$, $(x \wedge y)' = x' \vee y'$, $x \vee x' = 1$ and $x \wedge x' = 0$ for all $x, y \in L$.

2. Orthorings

First we introduce the concept of a generalized ortholattice and distinguish generalized ortholattices from other classes of lattices.

Definition 2.1. (cf., e.g., [15]). A lattice $(L; \vee, \wedge, 0)$ with 0 is called a *sectionally complemented lattice* if for each $a \in L$, $([0, a]; \vee, \wedge)$ is a complemented lattice, i.e. for every $a, b \in L$ with $b \leq a$ there exists an element c of L with $c \leq a$, $b \vee c = a$ and $b \wedge c = 0$.

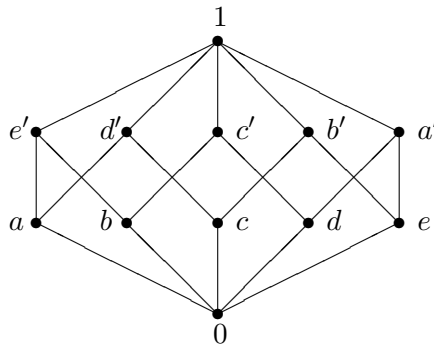
Definition 2.2. $(L; \vee, \wedge, ({}^a; a \in L), 0)$ where $(L; \vee, \wedge, 0)$ is a lattice with 0 is called a *generalized ortholattice* if for each $a \in L$, $([0, a]; \vee, \wedge, {}^a, 0, a)$ is an ortholattice.

Of course, if $(L; \vee, \wedge, ({}^a; a \in L), 0)$ is a generalized ortholattice, then $(L; \vee, \wedge, 0)$ is a sectionally complemented lattice.

Example 2.1. The five-element modular non-distributive lattice is sectionally complemented but it cannot be considered as a generalized ortholattice since it has an odd number of elements.

Example 2.2. If $(L; \vee, \wedge, ', 0, 1)$ is an orthomodular lattice, i.e. an ortholattice satisfying $y = x \vee (y \wedge x')$ for all $x, y \in L$ with $x \leq y$, then $(L; \vee, \wedge, (x \mapsto x' \wedge a; a \in L), 0)$ is a generalized ortholattice.

Example 2.3. The following Hasse diagram shows a non-orthomodular generalized ortholattice:



Next we introduce ring-like structures corresponding to generalized ortholattices.

Definition 2.3. An *orthoring* is an algebra $(R; +, \cdot, 0)$ of type $(2, 2, 0)$ satisfying

$$(O1) \quad x + y = y + x,$$

$$(O2) \quad x + 0 = x,$$

$$(O3) \quad xy = yx,$$

$$(O4) \quad (xy)z = x(yz),$$

$$(O5) \quad xx = x,$$

$$(O6) \quad x0 = 0,$$

$$(O7) \quad (xy + x) + x = xy,$$

$$(O8) \quad ((x + y) + xy) + xy = x + y,$$

$$(O9) \quad (xy + x)x = xy + x,$$

$$(O10) \quad (x + y)xy = 0,$$

$$(O11) \quad ((x + y) + xy)x = x,$$

$$(O12) \quad ((xy + xz) + xyz)x = (xy + xz) + xyz$$

and

$$(O13) \quad (xyz + x)(xy + x) = xy + x.$$

Remark 2.1. Orthorings $\mathcal{R} = (R; +, \cdot, 0)$ are of characteristic 2, i.e. $x + x = 0$ for all $x \in R$.

Proof.

$$\begin{aligned} x + x &\stackrel{(O1),(O2)}{=} (0 + x) + x \stackrel{(O4),(O10)}{=} ((x + x)xx + x) + x \stackrel{(O3)-(O5)}{=} \\ &\stackrel{(O3)-(O5)}{=} (x(x + x) + x) + x \stackrel{(O7)}{=} x(x + x) \stackrel{(O3)-(O5)}{=} (x + x)xx \stackrel{(O10)}{=} 0. \end{aligned}$$

■

Now we can state our main result describing a natural bijective correspondence between generalized ortholattices and orthorings.

Theorem 2.1. *For fixed set L the formulas*

$$x + y := (x \wedge y)^{x \vee y},$$

$$xy := x \wedge y$$

and

$$x \vee y := (x + y) + xy,$$

$$x \wedge y := xy,$$

$$x^y := x + y$$

induce mutually inverse bijections between the set of all generalized ortholattices on L and the set of all orthorings on L .

Proof. Let $\mathcal{L} = (L; \vee, \wedge, ({}^a; a \in L), 0)$ be a generalized ortholattice and put $x + y := (x \wedge y)^{x \vee y}$ and $xy := x \wedge y$ for all $x, y \in L$. Let $x, y, z \in L$. Then

$$(x + y) + xy = ((x \wedge y)^{x \vee y} \wedge x \wedge y)^{(x \wedge y)^{x \vee y} \vee (x \wedge y)} = 0^{x \vee y} = x \vee y,$$

$$x + 0 = 0^x = x,$$

$$(xy + x) + x = ((x \wedge y)^x)^x = x \wedge y = xy,$$

$$((x + y) + xy) + xy = (x \wedge y)^{x \vee y} = x + y,$$

$$(xy + x)x = (x \wedge y)^x \wedge x = (x \wedge y)^x = xy + x,$$

$$(x + y)xy = (x \wedge y)^{x \vee y} \wedge x \wedge y = 0,$$

$$((x + y) + xy)x = (x \vee y) \wedge x = x,$$

$$\begin{aligned} ((xy + xz) + xyz)x &= ((xy + xz) + (xy)(xz))x = ((x \wedge y) \vee (x \wedge z)) \wedge x = \\ &= (x \wedge y) \vee (x \wedge z) = (xy + xz) + (xy)(xz) = (xy + xz) + xyz \text{ and} \end{aligned}$$

$$(xyz + x)(xy + x) = (x \wedge y \wedge z)^x \wedge (x \wedge y)^x = (x \wedge y)^x = xy + x.$$

Hence, $(L; +, \cdot, 0)$ is an orthoring. Moreover,

$$(x + y) + xy = x \vee y,$$

$$xy = x \wedge y$$

and

$$x \leq y \text{ implies } x + y = x^y.$$

Therefore, the algebra induced by $(L; +, \cdot, 0)$ according to the formulas given in the theorem coincides with \mathcal{L} .

Conversely, let $\mathcal{R} = (L; +, \cdot, 0)$ be an orthoring and put $x \vee y := (x + y) + xy$, $x \wedge y := xy$ and $x^y := x + y$ for all $x, y \in L$. Let $(L; \leq)$ denote the poset corresponding to the meet-semilattice $(L; \cdot)$ and $x, y, z \in L$. Then

$$\begin{aligned}
x(x \vee y) &= x((x + y) + xy) \stackrel{(O3)}{=} ((x + y) + xy)x \stackrel{(O11)}{=} x, \text{ i.e. } x \leq x \vee y, \\
y(x \vee y) &= y((x + y) + xy) \stackrel{(O1),(O3)}{=} ((y + x) + yx)y \stackrel{(O11)}{=} y, \text{ i.e. } y \leq x \vee y, \\
x, y \leq z &\text{ implies } (x \vee y)z = ((x + y) + xy)z \stackrel{(O3),(O4)}{=} ((zx + zy) + zxy)z \\
&\stackrel{(O4),(O12)}{=} (zx + zy) + zxy \stackrel{(O3),(O4)}{=} (x + y) + xy = x \vee y, \text{ i.e. } x \vee y \leq z, \\
x \leq y &\text{ implies } x^y y = (x + y)y \stackrel{(O3)}{=} (yx + y)y \stackrel{(O9)}{=} yx + y \stackrel{(O3)}{=} x + y = x^y, \\
&\text{ i.e. } x^y \leq y, \\
x \leq y &\text{ implies } (x^y)^y = (x + y) + y \stackrel{(O3)}{=} (yx + y) + y \stackrel{(O7)}{=} yx \stackrel{(O3)}{=} x, \\
x \leq y \leq z &\text{ implies } y^z x^z = (y + z)(x + z) \stackrel{(O3),(O4)}{=} (zyx + z)(zy + z) \\
&\stackrel{(O4),(O13)}{=} zy + z \stackrel{(O3)}{=} y + z = y^z, \text{ i.e. } y^z \leq x^z \text{ and} \\
x \leq y &\text{ implies } x \wedge x^y = x(x + y) \stackrel{(O3),(O4)}{=} (x + y)xy \stackrel{(O10)}{=} 0.
\end{aligned}$$

Hence $(L; \vee, \wedge, ({}^a; a \in L), 0)$ is a generalized ortholattice. Moreover,

$$\begin{aligned}
(x \wedge y)^{x \vee y} &= xy + ((x + y) + xy) \stackrel{(O1)}{=} ((x + y) + xy) + xy \stackrel{(O8)}{=} x + y \text{ and} \\
x \wedge y &= xy.
\end{aligned}$$

This shows that the algebra induced by $(L; \vee, \wedge, ({}^a; a \in L), 0)$ according to the formulas of the theorem coincides with \mathcal{R} . ■

Remark 2.2. If $(L; \vee, \wedge, ', 0, 1)$ is a Boolean algebra, then $x + y = (x \wedge y)^{x \vee y}$ is the well-known symmetric difference since $(x \wedge y') \vee (x' \wedge y) = (x \wedge y)' \wedge (x \vee y) = (x \wedge y)^{x \vee y}$ for all $x, y \in L$.

Comparing the definition of an orthoring to the definition of a Boolean pseudoring introduced in [7] and comparing the definition of a generalized ortholattice with that of a generalized orthomodular lattice (cf. [17]), we obtain

Theorem 2.2. *An orthoring $(R; +, \cdot, 0)$ is a Boolean pseudoring if and only if $(x + y)x = x + xy$ and $(xyz + x)y = xyz + xy$ for all $x, y, z \in R$. A generalized ortholattice $(L; \vee, \wedge, ({}^a; a \in L), 0)$ is a generalized orthomodular lattice if and only if $x^z \wedge y = x^y$ for all $x, y, z \in L$ with $x \leq y \leq z$.*

Proof. The second assertion can be proved as follows: Let $\mathcal{L} = (L; \vee, \wedge, ({}^a; a \in L), 0)$ be a generalized ortholattice. If \mathcal{L} is a generalized orthomodular lattice, then $x^z \wedge y = x^y$ for all $x, y, z \in L$ with $x \leq y \leq z$ according to the definition of a generalized orthomodular lattice. Conversely, if $x^z \wedge y = x^y$ for all $x, y, z \in L$ with $x \leq y \leq z$, then $x \vee (y \wedge x^z) = x \vee x^y = y$ for all $x, y, z \in L$ with $x \leq y \leq z$ and, hence, $([0, z]; \vee, \wedge, {}^z, 0, z)$ is orthomodular for all $z \in L$. Therefore, \mathcal{L} is a generalized orthomodular lattice. ■

3. Congruence and ideal properties

For an overview on congruence conditions, their characterizations and the theory of ideals in universal algebras, see [4].

A variety is called *arithmetical* if it is both congruence permutable and congruence distributive. A variety with a constant term 0 is called *weakly regular* if any congruence of an algebra belonging to this variety is determined by its 0-class.

It is easy to see that the congruence lattice of an orthoring is a sublattice of the congruence lattice of the corresponding sectionally complemented lattice. Since it is well known that sectionally complemented lattices are arithmetical and weakly regular (see, e.g., [15]), this carries over to orthorings.

Here we will provide a different (and direct) proof of this result.

Theorem 3.1. *Orthorings are arithmetical and weakly regular.*

Proof. Consider the terms

$$\begin{aligned} t_1(x, y) &:= xy + x, \\ t_2(x, y) &:= xy + y, \\ t(x, y, z, u) &:= (y + u) + z \text{ and} \\ m(x, y, z) &:= (xy + yz) + zx. \end{aligned}$$

We show that t_1 , t_2 and t satisfy the identities

$$\begin{aligned} t_1(x, x) &= t_2(x, x) = 0, \\ t(x, y, t_1(x, y), t_2(x, y)) &= x \text{ and} \\ t(x, y, 0, 0) &= y \end{aligned}$$

from which it follows that orthorings are permutable and weakly regular according to Theorem 6.4.11 of [4]. Moreover, we prove that m is a majority term, i.e. it satisfies

$$m(x, x, y) = m(x, y, x) = m(y, x, x) = x$$

from which we obtain that ortholattices are congruence distributive according to Corollary 3.2.4 of [4].

The following calculations yield the desired identities:

$$\begin{aligned} (x + xy) + xy &= ((x \wedge y)^x \wedge (x \wedge y))^{(x \wedge y)^x \vee (x \wedge y)} = 0^x = x, \\ t_1(x, x) &= xx + x \stackrel{(O5)}{=} x + x = 0, \\ t_2(x, x) &= xx + x \stackrel{(O5)}{=} x + x = 0, \\ t(x, y, t_1(x, y), t_2(x, y)) &= (y + (xy + y)) + (xy + x) \stackrel{(O1),(O3)}{=} \\ &\stackrel{(O1),(O3)}{=} ((yx + y) + y) + (xy + x) \stackrel{(O7)}{=} yx + (xy + x) \stackrel{(O1),(O3)}{=} \\ &\stackrel{(O1),(O3)}{=} (x + xy) + xy = x, \end{aligned}$$

$$\begin{aligned}
 t(x, y, 0, 0) &= (y + 0) + 0 \stackrel{(O2)}{=} y, \\
 m(x, x, y) &= (xx + xy) + yx \stackrel{(O3),(O5)}{=} (x + xy) + xy = x, \\
 m(x, y, x) &= (xy + yx) + xx \stackrel{(O3),(O5)}{=} (xy + xy) + x \stackrel{(O1),(O2)}{=} x \text{ and} \\
 m(y, x, x) &= (yx + xx) + xy \stackrel{(O1),(O3),(O5)}{=} (x + xy) + xy = x. \quad \blacksquare
 \end{aligned}$$

Let \mathcal{V} be a variety with a constant term 0 , \mathcal{A} an algebra belonging to \mathcal{V} and B a subset of the carrier set of \mathcal{A} . A term $t(x_1, \dots, x_n)$ is called an *ideal term* with respect to the variables $x_i, i \in I$, ($I \subseteq \{1, \dots, n\}$) if $t(x_1, \dots, x_n) = 0$ whenever $x_i = 0$ for all $i \in I$. Let $t(x_1, \dots, x_n)$ be an ideal term with respect to the variables $x_i, i \in I$. B is called *closed with respect to t* if $t(a_1, \dots, a_n) \in B$ provided $a_1, \dots, a_n \in A$ and $a_i \in B$ for all $i \in I$. B is called an *ideal* of \mathcal{A} if B is closed with respect to all ideal terms. A set T of ideal terms is called a *basis of ideal terms* if a subset of the carrier set of an algebra \mathcal{C} belonging to \mathcal{V} is an ideal of \mathcal{C} whenever it is closed with respect to all ideal terms belonging to T . If Θ is a congruence on \mathcal{A} , then the congruence class $[0]\Theta$ of 0 with respect to Θ is called the *congruence kernel* of Θ . It is easy to see that every congruence kernel is an ideal. If \mathcal{V} has a so-called *subtractive term*, i.e. a binary term s satisfying $s(x, 0) = x$ and $s(x, x) = 0$, then, conversely, every ideal is a congruence kernel (cf. Theorems 6.6.11 and 10.1.10 of [4]). This is the case with orthorings, because the term $s(x, y) := x + y$ serves as a subtractive term.

Ideals in orthorings can now be characterized as follows:

Theorem 3.2. *A subset I of the base set R of an orthoring \mathcal{R} containing 0 is an ideal of \mathcal{R} if and only if $x, y, z, u \in R$ and $xy + x, xy + y, zu + z, zu + u \in I$ together imply $(x + z)(y + u) + (x + z), xyzu + xz \in I$.*

Proof. This follows from Theorem 10.3.1 of [4] by using the terms introduced in the proof of Theorem 3.1. ■

Corollary 3.1. *Every ideal I of an orthoring $\mathcal{R} = (R; +, \cdot, 0)$ is the kernel of the congruence $\Theta_I := \{(x, y) \in R \times R \mid xy + x \in I \text{ and } xy + y \in I\}$ on \mathcal{R} .*

Proof. It is almost evident that I is the kernel of Θ_I where Θ_I is reflexive and compatible. However, the variety of orthorings is congruence permutable according to Theorem 3.1, and by [19], every compatible reflexive relation on \mathcal{R} is a congruence on \mathcal{R} (see also Corollary 3.1.13 in [4]). ■

Finally, we present a finite basis of ideal terms for orthorings:

Theorem 3.3. *The following terms form a basis of ideal terms for orthorings:*

0,

$$(((x+y_1) + y_2) + ((z+y_3) + y_4))(x+z) + (((x+y_1) + y_2) + ((z+y_3) + y_4)),$$

$$(((x+y_1) + y_2) + ((z+y_3) + y_4))(x+z) + (x+z),$$

$$((x+y_1) + y_2)((z+y_3) + y_4)xz + ((x+y_1) + y_2)((z+y_3) + y_4),$$

$$((x+y_1) + y_2)((z+y_3) + y_4)xz + xz,$$

and

$$y_1 + y_2.$$

Proof. This follows from Theorem 10.3.4 of [4] by using the terms introduced in the proof of Theorem 3.1. ■

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