Discussiones Mathematicae
General Algebra and Applications 24 (2004) 125–135

CLIFFORD SEMIFIELDS

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Abstract

It is well known that a semigroup S is a Clifford semigroup if and only if S is a strong semilattice of groups. We have recently extended this important result from semigroups to semirings by showing that a semiring S is a Clifford semiring if and only if S is a strong distributive lattice of skew-rings. In this paper, we introduce the notions of Clifford semidomain and Clifford semifield. Some structure theorems for these semirings are obtained.

Keywords: skew-ring, Clifford semiring, Clifford semidomain, Clifford semifield, Artinian Clifford semiring.

2000 Mathematics Subject Classification: 16Y60, 20N10, 20M07, 12K10.

*The research is supported by CSIR, India.

1. Introduction

Recall that a *semiring* $(S;+,\cdot)$ is a type $(2,2)$ algebra whose semigroup reducts $(S;+)$ and $(S;\cdot)$ are connected by distributivity, that is, $a(b+c) = ab+ac$ and $(b+c)a = ba+ca$ for all $a,b,c \in S$. We call a semiring $(S;+,\cdot)$ *additive regular* if for every element $a \in S$ there exists an element $x \in S$ such that $a+x+a = a$. Additive regular semirings were first studied by J. Zeleznekow [7] in 1981. We call a semiring $(S;+,\cdot)$ an *additive inverse semiring* if $(S;+)$ is an additive inverse semigroup. Additive inverse semirings were first studied by Karvellas [3] in 1974. Throughout this paper, we always let $E^+(S)$ be the set of all additive idempotents of the semiring S. Also we denote the set of all inverse elements of a in the regular semigroup $(S;+)$ by $V^+(a)$.

We call an element a of a semiring $(S;+,\cdot)$ *completely regular* (see [6]) if there exists an element $x \in S$ such that

(i) $a+x+a = a$,

(ii) $a+x = x+a$

and

(iii) $a(a+x) = a+x$.

Naturally, we call a semiring $(S;+,\cdot)$ *completely regular* ([6]) if every element a of S is completely regular. The condition (iii) can be replaced by the condition

(iii') $(a+x)a = a+x$.

If $a \in S$ is completely regular, and (iii') is satisfied, then $y = x+a+x \in V^+(a)$ and the conditions (i), (ii) and (iii) hold. Moreover, $y = x+a+x \in V^+(a)$ is unique and is denoted by a'. Also we proved in [6] (cf. Lemmas 2.5-2.7) the following:

Theorem 1.1. *Let S be a completely regular semiring. Then for any $a,b \in S$ and $e \in E^+(S)$ we have*

(i) $(a')' = a$,

(ii) $ab' = (ab)' = a'b$,

(iii) $ab = a'b'$ and

(iv) $e' = e$ and $e^2 = e$.

■

Recall that an ideal I of a semiring S is a *k-ideal* of S if $a \in I$ and either $a + x \in I$ or $x + a \in I$ for some $x \in S$ implies $x \in I$. Also, an ideal I of a semiring S is called a *full ideal* if $E^+(S) \subseteq I$. Again, if I is a k-ideal of a semiring S, then the quotient semiring of S by I is denoted by S/I.

If S is a completely regular semiring as well as an additive inverse semiring, then $E^+(S)$ is an ideal of S but $E^+(S)$ may not be a k-ideal of S. For instance, let $S = \{0, a, b\}$ be a semiring with the following Cayley tables:

$+$	0	a	b
0	0	a	b
a	a	0	b
b	b	b	b

$\cdot$	0	a	b
0	0	0	0
a	0	0	0
b	0	0	b

.

Then we can easily see that the additive reduct $(S; +)$ is an additive inverse semigroup. It is also easy to see that $(S; +, \cdot)$ is a completely regular semiring because $a(a+a) = a0 = 0 = a+a$ and $b(b+b) = bb = b = b+b$ hold. In this example, $E^+(S) = \{0, b\}$ is clearly an ideal of S but since $a+b = b \in E^+(S)$ and $a \notin E^+(S)$, $E^+(S)$ is not a k-ideal of S.

In view of the above example, we call a completely regular semiring S a *Clifford semiring* if S is an additive inverse semiring such that $E^+(S)$ forms a distributive lattice as well as a k-ideal of S.

According to M.P. Grillet [2], a semiring $(S; +, \cdot)$ is called a *skew-ring* if its additive reduct $(S; +)$ is a group.

Definition 1.2. Let D be distributive lattice and $\{S_\alpha : \alpha \in D\}$ be a family of pairwise disjoint semirings which are indexed by the elements of D. For each $\alpha \leq \beta$ in D, we now embed S_α in S_β via a semiring monomorphism $\phi_{\alpha,\beta}$ satisfying the following conditions

(1.1) $\phi_{\alpha,\alpha} = I_{S_\alpha}$, the identity mapping on S_α

(1.2) $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ if $\alpha \leq \beta \leq \gamma$

(1.3) $S_\alpha\phi_{\alpha,\gamma}S_\beta\phi_{\beta,\gamma} \subseteq S_{\alpha\beta}\phi_{\alpha\beta,\gamma}$ if $\alpha + \beta \leq \gamma$

On $S = \bigcup_{\alpha \in D} S_\alpha$ we define addition $+$ and multiplication $\cdot$ for $a \in S_\alpha, b \in S_\beta$, as follows

(1.4) $a + b = a\phi_{\alpha,\alpha+\beta} + b\phi_{\beta,\alpha+\beta}$

and $a \cdot b = c \in S_{\alpha\beta}$ such that (1.5) $c\phi_{\alpha\beta,\alpha+\beta} = a\phi_{\alpha,\alpha+\beta} \cdot b\phi_{\beta,\alpha+\beta}$.
Like the notation of strong semilattice of semigroups, we denote the above system by $S =< D, S_\alpha, \phi_{\alpha,\beta} >$ and call it the *strong distributive lattice D of the semirings $S_\alpha, \alpha \in D$.*

In our paper [5], we have proved the following theorem.

Theorem 1.3. *A semiring S is a Clifford semiring if and only if S is a strong distributive lattice of skew-rings.* ∎

By using Theorem 1.3, we see at once that if S is additive commutative, then S is a Clifford semiring if and only if S is strong distributive lattice of rings.

In this paper, we introduce the notions of Clifford semidomain and Clifford semifield. We show that any Artinian semidomain is a Clifford semifield. Also we prove that a Clifford semiring S with 1 and 0 is k-ideal free if and only if S is a field or $S = \{0, 1\}$.

2. Clifford semifields

Throughout the paper, we let S denote a semiring with commutative addition. We first introduce the concept of Clifford semidomain and Clifford semifield.

Definition 2.1. Let S be a semiring with $E^+(S) \neq \phi$. We say that S *is without additive idempotent divisors* if for any $a, b \in S, ab \in E^+(S)$ implies either $a \in E^+(S)$ or $b \in E^+(S)$. Otherwise we say that S *has additive idempotent divisors.*

Definition 2.2. Let S be a Clifford semiring with 1 such that $1 \notin E^+(S)$. A non additive idempotent element $a \in S$ is said to be *left invertible* if there exists an element $r \in S$ such that $ra + 1 + 1' = 1$. In this case, r is called the *left inverse* of a. Similarly, we can define *right invertible element* in a Clifford semiring. An element is said to be *invertible* if it is left invertible as well as right invertible. If a is invertible, we say that a is a *unit* in S.

Definition 2.3. A Clifford semiring S is called a *Clifford semidomain* if

(i) $1 \in S$ such that $1 \notin E^+(S)$,

(ii) S is multiplicative commutative

and

(iii) S does not contain any additive idempotent divisor.

Example 2.4. Let R be an integral domain with an identity 1_R and D be a distributuve lattice with a greatest element 1_D. Then $R \times D$ is a Clifford semidomain.

Definition 2.5. A Clifford semiring S is called a *Clifford semifield* if

(i) $1 \in S$ such that $1 \notin E^+(S)$,

(ii) S is multiplicative commutative

and

(iii) every non additive idempotent element of S is a unit.

Example 2.6. Let F be a field and D be a distributuve lattice with a greatest element 1_D. Then $F \times D$ is a Clifford semifield.

Definition 2.7. An ideal P of a semiring S is called a *prime ideal* of S if for any two ideals A, B of S such that $AB \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$.

Proposition 2.8. *Let S be a Clifford semiring such that $(S, \cdot)$ is commutative. Then an ideal P is prime if and only if $ab \in P$ implies either $a \in P$ or $b \in P$.*

The proof is similar to a characterizations of prime ideals in semigroups and we omit the proof. ∎

Definition 2.9. An ideal M of a semiring S is called a *maximal ideal* of S if there exists no ideal I of S such that $M \subsetneq I \subsetneq S$.

It is easy to verify the following lemma:

Lemma 2.10. *Let S be a Clifford semiring. Then any maximal ideal of S is a prime ideal.* ∎

We now prove the following theorem:

Theorem 2.11. *Let S be a Clifford semiring with 1 such that $(S, \cdot)$ is commutative. Then a k-ideal P is a prime ideal if and only if S/P is a Clifford semidomain.*

Proof. First suppose that a k-ideal P is prime. Let $a+P, b+P \in S/P$ be such that $(a+P)(b+P) \in E^+(S/P)$. Then $ab \in P$. Since P is prime either $a \in P$ or $b \in P$. So either $a+P \in E^+(S/P)$ or $b+P \in E^+(S/P)$. Thus, S/P has no additive idempotent divisor. This proves that S/P is a Clifford semidomain.

Conversely, let a k-ideal P be such that S/P is a Cliffod semidomain. Let $a, b \in S$ be such that $ab \in P$. Then $ab+P \in E^+(S/P)$, i.e., $(a+P)(b+P) \in E^+(S/P)$. Since S/P is a Clifford semidomain, so either $a+P \in E^+(S/P)$ or $b+P \in E^+(S/P)$, i.e., either $a \in P$ or $b \in P$. Thus, P is a prime ideal of S. ■

By the definition of Clifford semifield, we now prove the following theorem.

Theorem 2.12. *Let S be a Clifford semiring with 1 such that $(S, \cdot)$ is commutative. Then a k-ideal M is a maximal ideal if and only if S/M is a Clifford semifield.*

Proof. First we suppose that a k-ideal M is maximal. Let $a+M \notin E^+(S/M)$. Then $a \notin M$. Let $M' =< M, a >$, where $< M, a >$ denotes the ideal of S generated by M and a. Then $M \subsetneq M'$. Since M is maximal, $M' = S$. Thereby, we have $1 = m + sa$ for some $m \in M$ and $s \in S$. This leads to $1+M = (m+M)+(sa+M) = ((m+m')+M)+(sa+M)$. Hence, $1+M = (sa+M)+((1+1')+M)$, i.e., $(s+M)(a+M)+(1+M)+(1'+M) = 1+M$. This means that $a+M$ is invertible in S/M and hence S/M is a Clifford semifield.

Conversely, let M be a k-ideal so that S/M is a Cliffod semifield. Let $M \subsetneq I \subseteq S$ be an ideal of S. Then there exists an element $a \in I$ such that $a \notin M$. This leads to $a+M \notin E^+(S/M)$ and hence there exists an element $s+M \in S/M$ such that $(s+M)(a+M)+(1+M)+(1'+M) = 1+M$, i.e., $sa+1+1'+1' \in M$. This implies that $sa+1' \in M$, i.e., $1+s'a \in M \subseteq I$. Also, $a \in I$ implies $sa \in I$, and thereby, we have $1 = 1+s'a+sa \in I$. Hence, we have $I = S$ and this shows that M is a maximal ideal of S. ■

3. Artinian Clifford semiring

Definition 3.1. A Clifford semiring S is called *Artinian Clifford semiring* if any descending chain of full ideals of S terminates, i.e. for any descending chain of full ideals $I_1 \supseteq I_2 \supseteq ....$ there exists a positive integer n such that $I_n = I_{n+1} = I_{n+2} = ...$

Example 3.2. Let R be a Artinian ring and $D = \{0, 1\}$ be the two element distributuve lattice. Then $F \times D$ is an Artinian Clifford semiring.

We can easily prove that a semiring S is Artinian if and only if any non empty collection of full ideals contains a minimal element. One can also easily verify that the homomorphic image of an Artinian Clifford semiring is again Artinian Clifford.

We first prove two lemmas.

Lemma 3.3. *Let S be an Artinian Clifford semiring with* 1*. Then S has a finite number of maximal full ideals.*

Proof. Suppose if possible that there exists an infinite sequence $\{M_i\}$ of distinct maximal full ideals of S. Then we consider the following descending chain of full ideals $M_1 \supseteq M_1M_2 \supseteq M_1M_2M_3 \supseteq \dots$

Since S is Artinian, there exists a positive integer n such that $M_1M_2...M_n$ $= M_1M_2...M_{n+1}$. Consequently, we have $M_1M_2...M_n \subseteq M_{n+1}$ and whence $M_k \subseteq M_{n+1}$ for some $k \leq n$ [by Lemma 2.10]. But since M_k is maximal ideal of S, we have $M_k = M_{n+1}$. This contradicts to the fact that M_i are all distinct. Hence, we obtain the required result. ∎

Lemma 3.4. *Every prime ideal of a Clifford semiring S with* 1 *is a k-ideal S.*

Proof. Let S be a Clifford semiring with 1 and P be a prime ideal of S. Let $a, a+b \in P$. We prove that $b \in P$. Since $a, a+b \in P$, we have $a'+a+b \in P$. This leads to, $b(a'+a)+b^2 \in P$, i.e. $b^2 \in P$. Since P is prime, this shows that $b \in P$. Hence, P is a k-ideal of S. ∎

The converse of the above lemma does not hold in general. For instance, we consider the following example.

Example 3.5. Let R be a ring. Then any ideal I of R is a k-ideal of R but not a prime ideal of R.

From Theorem 2.10. and Lemma 3.4, it immediately follows that, every maximal ideal of a Clifford semiring S with 1 is a k-ideal of S.

Definition 3.6. Let S be a semiring and A be non-empty subset of S. Then we call the set $\overline{A} = \{x \in S : x + a = b \text{ for some } a, b \in S\}$ the *k-closure* of A.

Proposition 3.7. *If S is a semisimple Artinian Clifford semiring with 1, then S is a k-closure of sum of finite number of proper k-ideal of S.*

Proof. Since S is Artinian Clifford semiring, S has a finite number of maximal full ideals. Let $M_1, M_2, ..., M_n$ be the finite number of maximal full ideals of S such that $\bigcap_{i=1}^{n} M_i = E^+(S)$ but $I_i = \bigcap_{\substack{k=1 \\ k \neq i}}^{n} M_k \neq E^+(S)$ for every i. Because each M_i is full maximal ideal of S, we see that each M_i is k-ideal and so is each I_i. Since M_i is maximal, we have $I_i + M_i = S$ for every i and $I_i \cap M_i = E^+(S)$.

Now, $S = I_i + M_i$, so we have, for $a \in S$, $a = x_i + y_i$, where $x_i \in I_i$ and $y_i \in M_i, i = 1, 2, ..., n$. This leads to $a + x'_k = x_k + x'_k + y_k \in M_k$ and $x'_i = 1'x_i \in I_i \subseteq M_k$ for $i \neq k$. Thus $a + \sum_{i=1}^{n} x'_i \in \bigcap_{i=1}^{n} M_i = E^+(S)$. Consequently, we have $a + \sum_{i=1}^{n} x'_i = e$ for some $e \in E^+(S)$. Now since $\sum_{i=1}^{n} x_i \in I_1 + I_2 + ... + I_n$ and $e = e + e + ... + e \in I_1 + I_2 + ... + I_n$, we see that $a \in \overline{I_1 + I_2 + ... + I_n}$. Hence, we have that $S \subseteq \overline{I_1 + I_2 + ... + I_n}$. The reverse inclusion is obvious and consequently, $S = \overline{I_1 + I_2 + ... + I_n}$. ■

Definition 3.8. Let S be a Clifford semiring. We define a relation θ on S by $\theta = \{(a, b) \in S \times S : a + b' \in E^+(S)\}$. One can easily verify that θ is a congruence relation on S such that S/θ is a ring.

Let S be a Clifford semidomain. Then S/θ is an integral domain, where θ is defined in Definition 3.8. Conversely, if S is an additive inverse semiring such that $E^+(S)$ is a k-ideal of S and S/θ is an integral domain, then S may not be a Clifford semiring. This follows from the following example.

Example 3.9. Let R be an integral domain and Y be a semiring which is not a distributive lattice but $(Y, +)$ is a band. Then the semiring $S = R \times Y$ is an additive inverse semiring such that $E^+(S) = \{0\} \times Y$ is a k-ideal of S, where 0 is the zero of the integral domain R. In this semiring, one can easily see that S/θ is is an integral domain but $E^+(S)$ is not a distributive lattice of S. Hence, S is not a Clifford semiring.

We now formulate an important theorem. This theorem characterizes the Clifford semidomain.

Theorem 3.10. *If S is a Clifford semidomain, then S is, up to the isomorphism, a subdirect product of an integral domain and a distributive lattice with a greatest element.*

Proof. Let S be a Clifford semidomain. Then S is a Clifford semiring and hence S is a strong distributive lattice D of rings R_α, $\alpha \in D$. Clearly, D is a bounded distributive lattice with a greatest element. Again since S is a Clifford semidomain, one can easily show that S/θ is an integral domain, where θ is defined in Definition 3.8.

We now define a mapping $\psi : S \to S/\theta \times D$ by $a\psi = (a\theta, \alpha)$, $a \in R_\alpha$. We can easily see that ψ is a monomorphism. Also the projection homomorphisms map $S\psi$ onto S/θ and D. Thus S is isomorphic to a subdirect product of an integral domain and a distributive lattice. ∎

Theorem 3.11. *Any Artinian semidomain (Clifford semidomain and Artinian Clifford semiring) is a Clifford semifield.*

Proof. To complete the proof, it suffices to prove that every non additive idempotent in S is a unit. For this purpose, we let $a \in S$ be such that $a \notin E^+(S)$. We consider the descending chain of full ideals $E^+(S) + Sa \supseteq E^+(S) + Sa^2 \supseteq E^+(S) + Sa^3 \supseteq \ldots$.

Since S is an Artinian semidomain, there exists a positive integer n such that $E^+(S)+Sa^n = E^+(S)+Sa^{n+1}$. Now, it is clear that $a^n \in E^+(S)+Sa^n$ and therefore there exists $e \in E^+(S)$ and $s \in S$ such that $a^n = e + sa^{n+1}$, i.e., $e+sa^{n+1}+(a^n)' = a^n+(a^n)'$. This leads to $e+(sa+1')a^n = a^n+(a^n)' = a^{n-1}(a + a') = a + a'$. Clearly, $a + a', e \in E^+(S)$ and $E^+(S)$ is a k-ideal of S. Hence, $(sa + 1')a^n \in E^+(S)$. Because S does not contain any additive idempotent divisor of S and $a \notin E^+(S)$, we must have $sa + 1' \in E^+(S)$. This leads to $sa + 1' = f$ for some $f \in E^+(S)$. Hence, we deduce that $sa+1+1' = 1+f = 1$ and consequently a is left invertible so that a is unit of S. This proves that S is a Clifford semifield. ∎

Theorem 3.12. *If S is an Artinian Clifford semiring, then every proper prime ideal of S is a maximal ideal.*

Proof. Let P be any proper prime ideal of S. Then P is a k-ideal of S and S/P is a Clifford semidomain. Moreover, S/P is an Artinian Clifford semiring. Hence, by Theorem 3.11, S/P is a Clifford semifield. Consequently, P is a maximal ideal of S. ■

The proof of the next Proposition is similar to the proof of Theorem 3.10. So, we omit the proof.

Proposition 3.13. *If S is a Clifford semifield, then S is, up to the isomorphisms, a subdirect product of a field and a distributive lattice with a greatest element.* ■

Recall that a semiring S is *full ideal free* if S has only two ideals, namely, $E^+(S)$ and the semiring S itself. Also, a semiring S with 0 is *k-ideal free* if S has only two k-ideals, namely, the ideal $\{0\}$ and the semiring S itself.

Finally, we prove the following two theorems.

Theorem 3.14. *A multiplicative commutative Clifford semiring S with 1 is a Clifford semifield if and only if S is full ideal free.*

Proof. First suppose that S is a Clifford semifield and I be an ideal of S such that $E^+(S) \subsetneq I$. Then there exists an element $a \in I$ such that $a \notin E^+(S)$. Now for $a \in S, a \notin E^+(S)$, there exists an element $r \in S$ such that $ar+1+1' = 1$. Now $ar \in I$ and also $1+1' \in I$. Thus, $1 = ar+1+1' \in I$ and hence $I = S$.

Conversely, let S be a Clifford semiring which is full ideal free. Let $a \in S$ be such that $a \notin E^+(S)$. Now $Sa + E^+(S)$ is an ideal of S such that $E^+(S) \subsetneq Sa + E^+(S)$. So $Sa + E^+(S) = S$. Hence, $1 = ra + e$ for some $r \in S$ and $e \in E^+(S)$. Then $1 = 1 + 1' + 1 = ra + e + 1' + 1 = ra + 1 + 1'$. Thus a is unit in S and consequently, S is a Clifford semifield. ■

Theorem 3.15. *An additive commutative and multiplicative commutative Clifford semiring S with 1 and 0 is k-ideal free if and only if S is a field or $S = \{0, 1\}$.*

Proof. First suppose that S is a k-ideal free. Now $E^+(S)$ is a k-ideal of S. So either $E^+(S) = \{0\}$ or $E^+(S) = S$. Let $E^+(S) = \{0\}$. Then S is a ring with 1. Let $a \in S$ be such that $a \neq 0$. Then Sa is a k-ideal of S. Hence, $Sa = S$ and thus we get $1 = ta$ for some $t \in S$. Consequently, S is a field.

Next, let $E^+(S) = S$. Then every element of S is additive idempotent and, hence, multiplicative idempotent. Now, Sa is a non-zero ideal of S for every $a(\neq 0) \in S$. Let $ra+b = ta$ for some $r, t \in S$. Then $a+ra+b = a+ta$, i.e., $a+b = a$. Therefore, $ba + b^2 = ba$, i.e., $ba + b = ba$. Then $b = ba \in Sa$. Hence, Sa is a k-ideal of S. Thus $Sa = S$ and it follows that, $ta = 1$ for some $t \in S$ i.e., $ta^2 = a$. Then $ta = a$ i.e., $a = 1$. Consequently, $S = \{0, 1\}$.

Converse is obvious. ■

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Received 31 December 2003
Revised 12 July 2004