

## ISOMORPHISMS OF DIRECT PRODUCTS OF LATTICE-ORDERED GROUPS

JÁN JAKUBÍK\*

*Matematický ústav SAV*  
*Grešákova 6, 040 01 Košice, Slovakia*

**e-mail:** kstefan@saske.sk

### Abstract

In this paper we investigate sufficient conditions for the validity of certain implications concerning direct products of lattice-ordered groups.

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### 1. INTRODUCTION

Let  $A, B$  and  $C$  be algebras of the same type;  $A \times B$  denotes the direct product of  $A$  and  $B$ .

The content of Section 5.7 (pp. 319–336) of the monograph [18] by McKenzie, McNulty and Taylor consists in investigating the conditions under which the following implications are valid:

- (1)  $A \times C \simeq B \times C \Rightarrow A \simeq B$ ;
- (2)  $A^n \simeq B^n \Rightarrow A \simeq B$  (where  $n$  is a positive integer);
- (3)  $A \simeq A \times B \times C \Rightarrow A \simeq A \times B$ .

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In the case of (1) and (2), finite algebras (or finite relational structures) have been taken into account; in some of the results dealing with (3), the algebras under consideration can be infinite. Cf. also [17] by McKenzie, [1] by Appleson and Lovász, and [15], [16] by Lovász.

It is remarked in [18] (p. 333) that the relation (3) resembles the theorem of Cantor-Bernstein of the set theory and that the situations, where an analogous result holds for categories, are extremely rare.

In fact, condition (1) implies condition (2) in any class of algebras, and condition (1) also implies condition (3) in any class of algebras that contain a one-element algebra.

In the present paper we consider the implications (1), (2) and (3) for lattice-ordered groups. We recall that a *lattice-ordered group* is an algebraic system  $(G; +, \wedge, \vee)$  such that  $(G; +)$  is a group,  $(G; \wedge, \vee)$  is a lattice and for each  $a, b, x, y \in G$  the relation

$$a \leq b \Rightarrow x + a + y \leq x + b + y$$

is valid.

Specker lattice-ordered groups were investigated by Conrad and Darnel [5]; for the definition, cf. Section 2 below.

We remark that if  $G$  is a lattice-ordered group, then either  $G$  is a one-element set or  $G$  is infinite.

We denote by  $\mathcal{G}$  the class of all lattice-ordered groups. Let  $\mathcal{G}'$  be a nonempty class of lattice-ordered groups which is closed with respect to isomorphisms and let  $i \in \{1, 2, 3\}$ . We say that the implication (i) holds in  $\mathcal{G}'$  if this implication is satisfied whenever the corresponding lattice-ordered groups standing in (i) belong to  $\mathcal{G}'$ .

Let  $\mathcal{G}_1$  be the class of all lattice-ordered groups  $G$  such that either  $G = \{0\}$  or  $G$  can be expressed as a direct product of directly indecomposable factors.

Further, let  $\mathcal{G}_2$  be the class of all  $G \in \mathcal{G}_1$  such that, if

$$G \simeq \prod_{i \in I} G_i,$$

where all  $G_i$  are directly indecomposable, then for each  $i \in I$  the set

$$\bar{i} = \{i(1) \in I : G_{i(1)} \simeq G_i\}$$

is finite.

We show that the implication (1) fails to be valid in  $\mathcal{G}_1$  and that this implication is valid in  $\mathcal{G}_2$ . If  $C \in \mathcal{G}_1$  and (1) is valid for each  $A, B \in \mathcal{G}_1$ , then  $C \in \mathcal{G}_2$ .

There exist Specker lattice-ordered groups  $A$  and  $B$  such that  $A^2 \simeq B^2$  and  $A \not\simeq B$ ; hence (2) is not valid in  $\mathcal{G}$ . We prove that (2) holds for  $\mathcal{G}_1$ .

Further, the implication (3) does not hold in  $\mathcal{G}$ . From the results of [13] it follows that this implication is valid in the class of all orthogonally  $\sigma$ -complete lattice-ordered groups.

The implication (1) for unary algebras was studied by Novotný in [19] and by Ploščica and Zelina in [20]. The implications (1) and (2) for monounary algebras have been dealt with by Jakubíková-Studenovská in [14].

The more detailed references related to (3) are given in Section 3 below.

## 2. PRELIMINARIES

For lattice-ordered groups we apply the notation as in [2] by Conrad.

Let  $G \in \mathcal{G}$ . We say that  $G$  is *directly indecomposable* if  $G \neq \{0\}$  and if, whenever  $G \simeq A \times B$ , then either  $A$  or  $B$  is a one-element set.

It is well-known that if

$$G \simeq \prod_{i \in I} G_i \quad \text{and} \quad G \simeq \prod_{j \in J} H_j$$

where all  $G_i$  and all  $H_j$  are directly indecomposable, then there exists a bijection  $\varphi$  of  $I$  onto  $J$  such that  $G_i \simeq H_{\varphi(i)}$  for each  $i \in I$ .

We express this fact by saying that if a lattice-ordered group has a direct decomposition with directly indecomposable factors, then this direct decomposition is unique up to isomorphisms.

Let  $n$  be a positive integer and  $A_1, A_2, \dots, A_n$  be elements of  $\mathcal{G}_1$ . Then there exists a set  $S = S(A_1, \dots, A_n)$  of lattice-ordered groups such that

- (i) if  $B \in S$ , then  $B$  is directly indecomposable and there exists  $n(1) \in \{1, 2, \dots, n\}$  such that  $B$  is isomorphic to a direct factor of  $A_{n(1)}$ ;
- (ii) if  $n(1) \in \{1, 2, \dots, n\}$  and  $X$  is a directly indecomposable direct factor of  $A_{n(1)}$ , then there exists  $B \in S$  with  $B \simeq X$ ;
- (iii) if  $B_1$  and  $B_2$  are distinct elements of  $S$ , then they are not isomorphic.

Let  $A \in \mathcal{G}$  and let  $\alpha$  be a nonzero cardinal. Let

$$G = \prod_{i \in I} G_i,$$

where  $\text{card}(I) = \alpha$  and  $G_i = A$  for each  $i \in I$ . Then we write  $G = A^\alpha$ . For  $\alpha = 0$  we consider  $A^\alpha$  to be the one-element lattice-ordered group  $\{0\}$ .

Let  $G \in \mathcal{G}$ . A nonempty subset  $Y$  of  $G$  is called *orthogonal* if  $Y \subseteq G^+$  and if  $y_1 \wedge y_2 = 0$  whenever  $y_1$  and  $y_2$  are distinct elements of  $Y$ . We say that  $G$  is orthogonally  $\sigma$ -complete if each denumerable orthogonal subset of  $G$  has the supremum in  $G$ .

Specker lattice-ordered groups have been dealt with by Conrad and Darnel in [3], [4], [5], and by the author in [12]. We recall the corresponding definition.

Let  $G \in \mathcal{G}$ . We denote by  $S_1(G)$  the set of all elements  $0 < x \in G$  such that the interval  $[0, x]$  of  $G$  is a Boolean algebra. The set  $S_0(G) = S_1(G) \cup \{0\}$  with the induced partial order is a generalized Boolean algebra.  $G$  is called a *Specker lattice-ordered* group if it is generated as a group by the set  $S_1(G)$ . All Specker lattice-ordered groups are abelian.

### 3. THE IMPLICATIONS (1), (2) AND (3)

We apply the notation as in Section 1.

Let  $Z$  be the additive group of all integers with the natural linear order. Let  $\alpha$  be an infinite cardinal. Put

$$A = Z, \quad B = Z \times Z, \quad C = Z^\alpha.$$

Then we have  $A \times C \simeq C \simeq B \times C$  and  $A \not\simeq B$ . Hence (1) is not valid in the class  $\mathcal{G}$ . Since  $Z$  is directly indecomposable, we conclude that (1) fails to be valid in  $\mathcal{G}_1$  as well.

Now assume that  $A, B$  and  $C$  belong to the class  $\mathcal{G}_2$ . Denote  $S = S(A, B, C)$ . If  $S = \emptyset$ , then we have  $A = B = C = \{0\}$  and then (1) obviously holds. Assume that  $S \neq \emptyset$ . We express  $S$  in the form

$$S = \{G_j\}_{j \in J}$$

such that for distinct elements  $j(1), j(2)$  of  $J$  we have  $G_{j(1)} \neq G_{j(2)}$ .

Then we can write

$$A \simeq \prod_{j \in J} G_j^{\alpha(a,j)}, B \simeq \prod_{j \in J} G_j^{\alpha(b,j)}, C \simeq \prod_{j \in J} G_j^{\alpha(c,j)},$$

where, for each  $j \in J$ ,  $\alpha(a, j)$ ,  $\alpha(b, j)$  and  $\alpha(c, j)$  are non-negative integers. Hence we have

$$A \times C \simeq \prod_{j \in J} G_j^{\alpha(a,j) + \alpha(c,j)},$$

$$B \times C \simeq \prod_{j \in J} G_j^{\alpha(b,j) + \alpha(c,j)}.$$

Assume that  $A \times C \simeq B \times C$ . Since the direct product of  $A \times C$  with directly indecomposable factors is unique up to isomorphisms (cf. Section 2), we conclude that for each  $j \in J$  the relation

$$\alpha(a, j) + \alpha(c, j) = \alpha(b, j) + \alpha(c, j)$$

is valid; thus  $\alpha(a, j) = \alpha(b, j)$ . Therefore  $A \simeq B$ .

Hence we have

**Theorem 3.1.** *The implication (1) holds for the class  $\mathcal{G}_2$ .* ■

By a modification of the method applied in the example above (concerning  $Z$ ), we obtain

**Proposition 3.2.** *Let  $C \in \mathcal{G}_1$ . Assume that for each  $A, B \in \mathcal{G}_1$  the implication (1) is valid. Then  $C \in \mathcal{G}_2$ .*

**Proof.** The case  $C = \{0\}$  is trivial; assume that  $C \neq \{0\}$ . Let  $S(C) = \{G_j\}_{j \in J}$ . We have

$$C \simeq \prod_{j \in J} G_j^{\alpha(c,j)},$$

where  $\alpha(c, j)$  is a nonzero cardinal for each  $j \in J$ . By way of contradiction, suppose that  $C$  does not belong to  $\mathcal{G}_2$ . Then there is  $j(1) \in J$  such that the cardinal  $\alpha(c, j(1))$  is infinite. Put  $A = G_{j(1)}, B = G_{j(1)} \times G_{j(1)}$ .

We have  $A, B \in \mathcal{G}_1$  and

$$A \times C \simeq C \simeq B \times C, \quad A \not\simeq C,$$

which is a contradiction.  $\blacksquare$

For dealing with the implication (2), we will apply the following results:

**Theorem 3.3** (cf. [7] by Hanf). *There exist Boolean algebras  $B_1$  and  $B_2$  such that  $B_1 \not\simeq B_2$  and  $B_1^2 \simeq B_2^2$ .*

**Proposition 3.4** (cf. [4] by Conrad and Darnel, Proposition 2.6). *Let  $B$  be a generalized Boolean algebra. There exists a Specker lattice-ordered group  $G$  such that  $B = S_0(G)$ .*

**Lemma 3.5** (cf. [12], Lemma 3.1). *Let  $G_1$  and  $G_2$  be Specker lattice-ordered groups such that  $S_0(G_1) \simeq S_0(G_2)$ . Then  $G_1 \simeq G_2$ .*

**Lemma 3.6.** *Let  $G_1$  and  $G_2$  be Specker lattice-ordered groups and  $H = G_1 \times G_2$ . Then  $H$  is a Specker lattice-ordered group and  $S_0(H) = S_0(G_1) \times S_0(G_2)$ .*

**Proof.** It is obvious that whenever  $z = (s_1, s_2)$  is an element of  $S_0(G_1) \times S_0(G_2)$ , then the interval  $[0, z]$  of  $H$  is a Boolean algebra, hence  $z \in S_0(H)$ . Conversely, let  $g = (g_1, g_2) \in S_0(H)$ . Thus  $[0, g]$  is a Boolean algebra, yielding that  $[0, g_i]$  ( $i = 1, 2$ ) are Boolean algebras as well; therefore  $g \in S_0(G_1) \times S_0(G_2)$ .

From the fact that the group  $G_i$  is generated by the set  $S_0(G_i)$  ( $i = 1, 2$ ), we conclude that the group  $H$  is generated by the set

$$\{(s_1, 0) : s_1 \in S_0(G_1)\} \cup \{(0, s_2) : s_2 \in S_0(G_2)\}$$

which is a subset of  $S_0(H)$ . Hence  $H$  is a Specker lattice-ordered group.  $\blacksquare$

Now, let  $B_1$  and  $B_2$  be as in Theorem 3.3. According to Proposition 3.4, there exist Specker lattice-ordered groups  $G_i$  with  $S_0(G_i) = B_i$ ; put  $H_i = G_i^2$  ( $i = 1, 2$ ). Then, by Lemma 3.6, we have

$$S_0(H_1) = B_1^2, \quad S_0(H_2) = B_2^2.$$

Thus  $S_0(H_1) \simeq S_0(H_2)$ . Hence Lemma 3.5 yields  $H_1 \simeq H_2$ , i.e.,  $G_1^2 \simeq G_2^2$ . On the other hand, since  $B_1 \not\simeq B_2$ , we get  $G_1 \not\simeq G_2$ .

We conclude that the implication (2) fails to be valid for the class  $\mathcal{G}$ .

**Theorem 3.7.** *The implication (2) is valid for the class  $\mathcal{G}_1$ .*

**Proof.** Let  $A, B \in \mathcal{G}_1$ . It suffices to consider the case  $A \neq \{0\} \neq B$ . Hence there exists a representation

$$A \simeq \prod_{i \in I} A_i^{\alpha(i)},$$

where  $\{A_i\}_{i \in I} = S(A)$ . Analogously, we have

$$B \simeq \prod_{j \in J} B_j^{\alpha(j)}$$

with  $\{B_j\}_{j \in J} = S(B)$ ; for each  $i \in I$  and  $j \in J$ ,  $\alpha(i)$  and  $\alpha(j)$  are nonzero cardinals. Let  $n$  be a positive integer; we obtain

$$A^n \simeq \prod_{i \in I} A_i^{n\alpha(i)}, \quad B^n \simeq \prod_{j \in J} B_j^{n\alpha(j)}.$$

Assume that  $A^n \simeq B^n$ . Since the direct product decompositions of  $A^n$  and of  $B^n$  into directly indecomposable factors are unique up to isomorphisms, we conclude that there exists a bijection  $\psi$  of  $I$  onto  $J$  such that for each  $i \in I$  we have

$$A_i \simeq B_{\psi(i)}, \quad n\alpha(i) = n\beta(\psi(i)).$$

From this we obtain  $\alpha_i = \beta_{\psi(i)}$ , whence  $A \simeq B$ . ■

Now let us consider the implication (3).

Let  $\mathcal{K}$  be a class of algebras of the same type which is closed with respect to isomorphism and with respect to direct products. We apply the standard terminology: if  $X, Y, Z \in \mathcal{K}$  and  $X \times Y \simeq Z$ , then  $X$  and  $Y$  are called *direct factors* of  $Z$ .

Consider the following condition for  $\mathcal{K}$ :

- (4) If  $P$  and  $Q$  are elements of  $\mathcal{K}$  such that
- (i)  $P$  is isomorphic to a direct factor of  $Q$ ,
  - (ii)  $Q$  is isomorphic to a direct factor of  $P$ ,

then  $P$  is isomorphic to  $Q$ .

**Lemma 3.8.** *The conditions (3) and (4) for the class  $\mathcal{K}$  are equivalent.*

**Proof.** Let (3) be valid for the class  $\mathcal{K}$ . Assume that  $P$  and  $Q$  belong to  $\mathcal{K}$  and that the conditions (i), (ii) from (4) are satisfied. Hence there exist  $X, Y \in \mathcal{K}$  such that

$$P \simeq Q \times X, \quad Q \simeq P \times Y.$$

Thus

$$Q \simeq Q \times X \times Y.$$

Then, in view of (3), we have  $Q \simeq Q \times X$ , hence  $Q \simeq P$ . Therefore, (4) holds for  $\mathcal{K}$ .

Conversely, assume that (4) is satisfied for  $\mathcal{K}$ . Let  $A, B, C \in \mathcal{K}$  and  $A \simeq A \times B \times C$ . Thus  $A \times B$  is a direct factor of  $A$ . Clearly,  $A$  is a direct factor of  $A \times B$ . In view of (4), we have  $A \simeq A \times B$ . Hence (3) is valid in  $\mathcal{K}$ . ■

Sikorski in [21] and Tarski in [23] (cf. also [22] by Sikorski) proved that (4) is valid for  $\sigma$ -complete Boolean algebras, generalizing the well-known Cantor-Bernstein theorem of the set theory.

Generalizations of Sikorski-Tarski Theorem for some types of lattice-ordered groups and of lattices have been proved by author (see [9], [11] and [13]).

There are related results for  $\sigma$ -complete  $MV$ -algebras (see [6] by De Simone, Mundici and Navara) and for orthogonally  $\sigma$ -complete pseudo  $MV$ -algebras (see [12] by the author).

We quote the following result from [13]:

**Theorem 3.9.** *Let  $\mathcal{K}$  be the class of all orthogonally  $\sigma$ -complete lattice-ordered groups. Then  $\mathcal{K}$  satisfies the condition (4).*

In view of Lemma 3.8, we have

**Corollary 3.10.** *Let  $\mathcal{K}$  be as in Theorem 3.9. Then the implication (3) is satisfied in  $\mathcal{K}$ .*

In [13] it has been shown by means of an example that the assumption of orthogonal  $\sigma$ -completeness cannot be omitted in Theorem 3.9. Hence  $\mathcal{G}$  does not satisfy the implication (3).

It is easy to verify that for lattice-ordered groups, (3) is a consequence of (1). (Cf. also the corresponding remark in Section 1.) Hence in view of 3.1, the implication (3) holds for the class  $\mathcal{G}_2$ .



Let  $\mathcal{K}$  be as in Theorem 3.9. Then  $\mathcal{K}$  is not a subclass of  $\mathcal{G}_2$ . In fact, if  $\alpha$  is an infinite cardinal, then the lattice-ordered group  $Z^\alpha$  belongs to  $\mathcal{K}$ , but  $Z^\alpha \notin \mathcal{G}_2$ .

A condition similar to (4) has been applied in [8] for dealing with complete lattice-ordered groups (instead of direct factors as in (4), convex  $\ell$ -subgroups have been considered).

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