

ADJOINTNESS BETWEEN THEORIES AND STRICT THEORIES

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Dedicated to Prof. Dr. habil. Klaus Denecke
on the occasion of his 60th birthday

Abstract

The categorical concept of a theory for algebras of a given type was founded by Lawvere in 1963 (see [8]). Hoehnke extended this concept to partial heterogeneous algebras in 1976 (see [5]). A partial theory is a *dhts*-category such that the object class forms a free algebra of type $(2,0,0)$ freely generated by a nonempty set J in the variety determined by the identities $ox \approx o$ and $xo \approx o$, where o and i are the elements selected by the 0-ary operation symbols.

If the object class of a *dhts*-category forms even a monoid with unit element I and zero element O , then one has a strict partial theory.

In this paper is shown that every J -sorted partial theory corresponds in a natural manner to a J -sorted strict partial theory via a strongly d -monoidal functor. Moreover, there is a pair of adjoint functors between the category of all J -sorted theories and the category of all corresponding J -sorted strict theories.

This investigation needs an axiomatic characterization of the fundamental properties of the category \underline{Par} of all partial function between arbitrary sets and this characterization leads to the concept of *dhts*- and *dht ∇ s*-categories, respectively (see [5], [11], [13]).

Keywords: symmetric monoidal category, *dhts*-category, partial theory, adjoint functor.

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1. INTRODUCTION

Heterogeneous algebras (many-sorted algebras) are, as well-known, algebraic systems consisting of a family of carrier sets and a family of functions such that their definition domain are cartesian products of certain carrier sets and their values are elements of a distinguished carrier set. The concept of such algebraic systems was independently introduced and investigated by P.J. Higgins ([4]) and G. Birkhoff & J.D. Lipson ([1]).

The development of a functorial semantic of algebraic theories for heterogeneous partial algebras requires a good knowledge about diagonal-halfterminal-halfdiagonal-inversional-symmetric monoidal categories (*dht ∇ s*-categories).

The morphism class of a category K will be denoted by K too, the object class of K by $|K|$, and the set of all morphisms in K out of an object A into an object B by $K[A, B]$.

The concept of a symmetric monoidal category in the sense of ([3]) is of fundamental importance.

Definition 1.1 ([3]). A sequence

$$K^\bullet = (K, \otimes, I, a, r, l, s)$$

is called *symmetric monoidal category*, if K is a category, $\otimes : K \times K \rightarrow K$ is a bifunctor, I is a distinguished object of K , $a = (a_{A,B,C} \in K[A \otimes (B \otimes C), (A \otimes B) \otimes C] \mid A, B, C \in |K|)$, $r = (r_A \in K[A \otimes I, A] \mid A \in |K|)$, $l = (l_A \in K[I \otimes A, A] \mid A \in |K|)$, $s = (s_{A,B} \in K[A \otimes B, B \otimes A] \mid A, B \in |K|)$ are families of isomorphisms in K (associativity, right-identity, left-identity, symmetry) such that

$$(F1) \quad \forall \rho, \rho' \in K \quad (\text{dom}(\rho \otimes \rho') = \text{dom} \rho \otimes \text{dom} \rho'),$$

$$(F2) \quad \forall \rho, \rho' \in K \quad (\text{cod}(\rho \otimes \rho') = \text{cod} \rho \otimes \text{cod} \rho'),$$

$$(F3) \quad \forall A, B \in |K| \quad (1_{A \otimes B} = 1_A \otimes 1_B),$$

$$(F4) \quad \forall A, B, C, A', B', C' \in |K| \quad \forall \rho \in K[A, B], \sigma \in K[B, C],$$

$$\rho' \in K[A', B'], \sigma' \in K[B', C'] \quad ((\rho \otimes \rho')(\sigma \otimes \sigma') = \rho\sigma \otimes \rho'\sigma'),$$

$$(M1) \quad \forall A, B, C, D \in |K|$$

$$(a_{A,B,C \otimes D} a_{A \otimes B, C, D} = (1_A \otimes a_{B,C,D}) a_{A, B \otimes C, D} (a_{A,B,C} \otimes 1_D)),$$

$$(M2) \quad \forall A, B \in |K| \quad (a_{A,I,B} (r_A \otimes 1_B) = 1_A \otimes l_B),$$

- (M3) $\forall A, B, C \in |K| (a_{A,B,C} s_{A \otimes B, C} a_{C, A, B} = (1_A \otimes s_{B, C}) a_{A, C, B} (s_{A, C} \otimes 1_B)),$
(M4) $\forall A, B \in |K| (s_{A, B} s_{B, A} = 1_{A \otimes B}),$
(M5) $\forall A \in |K| (s_{A, I} l_A = r_A),$
(M6) $\forall A, B, C, A', B', C' \in |K| \forall \rho \in K[A, A'], \sigma \in K[B, B'], \tau \in K[C, C']$
 $(a_{A, B, C} ((\rho \otimes \sigma) \otimes \tau) = (\rho \otimes (\sigma \otimes \tau)) a_{A', B', C'}),$
(M7) $\forall A, A' \in |K| \forall \rho \in K[A, A'] (r_{A'} \rho = (\rho \otimes 1_I) r_A),$
(M8) $\forall A, B \in |K| \forall \rho \in K[A, A'], \sigma \in K[B, B'] (s_{A, B} (\sigma \otimes \rho) =$
 $= (\rho \otimes \sigma) s_{A', B'}).$

A symmetric monoidal category is called *symmetric strictly monoidal*, if all associativity, right-identity, and all left-identity isomorphisms, are unit morphisms, i.e. identity morphisms in K (in the other terminology), only.

The defining conditions determine a lot of properties as follows.

Corollary 1.2. *Let K^\bullet be a symmetric monoidal category. Then*

- (M9) $\forall A, B \in |K| (a_{I, A, B} (l_A \otimes 1_B) = l_{A \otimes B}),$
(M10) $\forall A, B \in |K| (a_{A, B, I} r_{A \otimes B} = 1_A \otimes r_B),$
(M11) $r_I = l_I,$
(M12) $s_{I, I} = 1_{I \otimes I},$
(M13) $\forall A \in |K| (s_{I, A} r_A = l_A),$
(M14) $\forall A, A' \in |K| \forall \rho \in K[A, A'] (l_{A'} \rho = (1_I \otimes \rho) l_A),$
(ASR) $\forall A, B \in |K| (a_{A, B, I}^{-1} (1_A \otimes s_{B, I}) a_{A, I, B} = r_{A \otimes B} (r_A^{-1} \otimes 1_B)),$
(ASL) $\forall A, B \in |K| (a_{I, A, B} (s_{I, A} \otimes 1_B) a_{A, I, B}^{-1} = l_{A \otimes B} (1_A \otimes l_B^{-1})).$

Defining

- (B1) $b_{A, B, C, D} := a_{A \otimes B, C, D} (a_{A, B, C}^{-1} (1_A \otimes s_{B, C}) a_{A, C, B} \otimes 1_D) a_{A \otimes C, B, D}^{-1}$
for arbitrary $A, B, C, D \in |K|,$

one obtains furthermore

$$\begin{aligned}
\text{(B2)} \quad & \forall A, B, C, D \in |K| \quad (b_{A,B,C,D} = \\
& \quad = a_{A,B,C \otimes D}^{-1} (1_A \otimes a_{B,C,D} (s_{B,C} \otimes 1_D) a_{C,B,D}^{-1}) a_{A,C,B \otimes D}), \\
\text{(M15)} \quad & \forall A, B, C, D, A', B', C', D' \in |K| \quad \forall \rho \in K[A, A'], \sigma \in K[B, B'], \\
& \quad \lambda \in K[C, C'], \mu \in K[D, D'] \\
& \quad (b_{A,B,C,D} ((\rho \otimes \sigma) \otimes (\lambda \otimes \mu)) = ((\rho \otimes \lambda) \otimes (\sigma \otimes \mu)) b_{A',B',C',D'}), \\
\text{(M16)} \quad & \forall A, B, C, D \in |K| \quad (b_{A,B,C,D} b_{A,C,B,D} = 1_{A \otimes B} \otimes 1_{C \otimes D}), \\
\text{(M17)} \quad & \forall A, B, C, D \in |K| \quad (b_{A,B,C,D} (s_{A,C} \otimes s_{B,D}) = s_{A \otimes B, C \otimes D} b_{C,D,A,B}), \\
\text{(M18)} \quad & \forall A, A', B, B', C, C' \in |K| \\
& \quad (b_{(A,(B \otimes C)), A', (B' \otimes C')} (1_{A \otimes A'} \otimes b_{B,C,B',C'}) a_{A \otimes A', B \otimes B', C \otimes C'} \\
& \quad = (a_{A,B,C} \otimes a_{A',B',C'}) b_{(A \otimes B), C, (A' \otimes B'), C'} (b_{A,B,A',B'} \otimes 1_{C \otimes C'})), \\
& \text{or equivalently,} \\
& \quad \forall A, A', B, B', C, C' \in |K| \\
& \quad (a_{A \otimes A', B \otimes B', C \otimes C'} (b_{A,A',B,B'} \otimes 1_{C \otimes C'}) b_{(A \otimes B), (A' \otimes B'), C, C'} \\
& \quad = (1_{A \otimes A'} \otimes b_{B,B',C,C'}) b_{A,A', (B \otimes C), (B' \otimes C')} (a_{A,B,C} \otimes a_{A',B',C'})), \\
\text{(M19)} \quad & \forall A, B \in |K| \quad (b_{A,I,I,B} = 1_{A \otimes I} \otimes 1_{I \otimes B}), \\
\text{(M20)} \quad & \forall A, B \in |K| \quad (b_{A,I,B,I} = (r_A \otimes r_B) ((1_{A \otimes B} \otimes r_I) r_{A \otimes B})^{-1}), \\
\text{(M21)} \quad & \forall A, B \in |K| \quad (b_{I,A,I,B} = (l_A \otimes l_B) ((l_I \otimes 1_{A \otimes B}) l_{A \otimes B})^{-1}), \\
\text{(M22)} \quad & \forall A, B \in |K| \quad (b_{I,A,B,I} = s_{I \otimes A, B \otimes I} (s_{B,I} \otimes s_{I,A})), \\
\text{(M23)} \quad & \forall A, B \in |K| \quad (b_{A,B,I,I} = (1_{A \otimes B} \otimes r_I) r_{A \otimes B} (r_A^{-1} \otimes r_B^{-1})), \\
\text{(M24)} \quad & \forall A, B \in |K| \quad (b_{I,I,A,B} = (l_I \otimes 1_{A \otimes B}) l_{A \otimes B} (l_A^{-1} \otimes l_B^{-1})). \quad \blacksquare
\end{aligned}$$

Remark 1.3. By definition, the object class of a symmetric monoidal category K^\bullet forms an illegitimate algebra $(|K|, \otimes, I)$ of type $(2, 0)$, because the carrier is not a set.

Especially, of interest are objects consisting of finitely many factors I in arbitrary brackets, namely objects of the subalgebra $\langle I \rangle$ generated by the one element set $\{I\}$ as follows:

$$\begin{aligned}
\langle I \rangle^{(0)} & := \{I\}, \quad \langle I \rangle^{(n+1)} := \langle I \rangle^{(n)} \cup \{X \otimes Y \mid X, Y \in \langle I \rangle^{(n)}\}, \\
\langle I \rangle & := \bigcup_{n \in \mathbb{N}} \langle I \rangle^{(n)}.
\end{aligned}$$

This is in fact an algebra of type $(2, 0)$. The set $\langle I \rangle$ determines in a natural manner a symmetric monoidal subcategory $\langle I \rangle^\bullet$ of K^\bullet .

Moreover, every nonempty set $J \subseteq |K|$, $I \notin J$, determines a subalgebra H of type $(2, 0)$ as follows:

$$H^{(0)} := J \cup \{I\}, \quad H^{(n+1)} := H^{(n)} \cup \{X \otimes Y \mid X, Y \in H^{(n)}\},$$

$$H := \bigcup_{n \in \mathbb{N}} H^{(n)}.$$

The symmetric monoidal subcategory of K^\bullet generated by H , respectively by J , will be denoted by H^\bullet . Obviously, H^\bullet is a small category, since the carrier is a set.

If K^\bullet is a symmetric strictly monoidal category, then $(|K|, \otimes, I)$ is an illegitimate monoid, $\langle I \rangle$ is a one element set and every set J generates a monoid S with unit I .

Definition 1.4 ([10]). Let K^\bullet be a symmetric monoidal category. The monoidal subcategory \mathbf{C}_K^\bullet of K^\bullet generated by the morphism class

$$\begin{aligned} & \{1_X \mid X \in |K|\} \cup \{a_{X,Y,Z} \mid X, Y, Z \in |K|\} \cup \{r_X \mid X \in |K|\} \cup \{l_X \mid X \in |K|\} \\ & \cup \{a_{X,Y,Z}^{-1} \mid X, Y, Z \in |K|\} \cup \{r_X^{-1} \mid X \in |K|\} \cup \{l_X^{-1} \mid X \in |K|\} \end{aligned}$$

is called *central subcategory* of K^\bullet , its morphisms are called *central morphisms* of K^\bullet .

Remark 1.5. The class \mathbf{C}_K of all central morphisms of a symmetric monoidal category K^\bullet is given by the construction

$$\begin{aligned} \mathbf{C}_K^{(0)} := & \{1_X \mid X \in |K|\} \cup \{a_{X,Y,Z} \mid X, Y, Z \in |K|\} \cup \{r_X \mid X \in |K|\} \cup \{l_X \mid X \in |K|\} \\ & \cup \{a_{X,Y,Z}^{-1} \mid X, Y, Z \in |K|\} \cup \{r_X^{-1} \mid X \in |K|\} \cup \{l_X^{-1} \mid X \in |K|\}, \end{aligned}$$

$$\begin{aligned} \mathbf{C}_K^{(n+1)} := & \mathbf{C}_K^{(n)} \cup \{c_1 c_2 \mid c_1 \in K[X, Y] \wedge c_2 \in K[Y, P] \wedge c_1, c_2 \in \mathbf{C}_K^{(n)} \\ & \wedge X, Y, P \in |K|\} \cup \{c_1 \otimes c_2 \mid c_1, c_2 \in \mathbf{C}_K^{(n)}\}, \end{aligned}$$

$$\mathbf{C}_K = \bigcup_{n \in \mathbb{N}} \mathbf{C}_K^{(n)}$$

and forms a monoidal subcategory \mathbf{C}_K^\bullet of K^\bullet .

\mathbf{C}_K consists of unit morphisms only, if K^\bullet is symmetric strictly monoidal. The class of all unit morphisms of K is denoted by Un_K .

Coherence principle ([9], [6], [7]). *Let K^\bullet be a symmetric monoidal category. Then every planar closed diagram of central morphisms is commutative.*

Corollary 1.6. *Let K^\bullet be a symmetric monoidal category. Then, by the coherence principle, there is at most one central morphism $c_{X,Y} \in K$ between objects X and Y for every $X, Y \in |K|$. The central morphisms are isomorphisms only.*

Let X and Y be arbitrary objects of $\langle I \rangle^\bullet$. Then there is exactly one central morphism in the set $\langle I \rangle[X, Y]$.

The isomorphisms

$$i^{(n)} : I^n \rightarrow I \text{ and } i^{*(n)} : \bigotimes_{k=1}^n I \rightarrow I,$$

$$\text{where } I^n := \bigotimes_{k=1}^n I \text{ and } \bigotimes_{k=1}^1 I := I, \quad \bigotimes_{k=1}^{n+1} I := I \otimes \left(\bigotimes_{k=1}^n I \right),$$

between the different powers of I and the object I are expressable in the following form:

$$i^{(1)} = 1_I, \quad i^{(n+1)} = (i^{(n)} \otimes 1_I)r_I, \quad n \geq 1, \text{ especially } i^{(2)} = r_I,$$

$$i^{*(1)} = 1_I, \quad i^{*(n+1)} = (1_I \otimes i^{*(n)})l_I, \quad n \geq 1, \text{ especially } i^{*(2)} = l_I.$$

Proof. It remains to show the existence of an central morphism between arbitrary X and Y of $\langle I \rangle$.

a) One proves by induction over the complexity of X : $\forall X \in \langle I \rangle \exists c \in \langle I \rangle[X, I]$ ($c \in \mathbf{C}_K$):

$$\forall X \in \langle I \rangle^{(0)} (X = I \wedge 1_I \in \mathbf{C}_K);$$

$$\forall n \in \mathbb{N} [\forall X \in \langle I \rangle^{(n)} \exists c \in \langle I \rangle[X, I] (c \in \mathbf{C}_K) \Rightarrow$$

$$\Rightarrow \forall X \in \langle I \rangle^{(n+1)} \exists c \in \langle I \rangle[X, I] (c \in \mathbf{C}_K)],$$

since

$$\begin{aligned} \forall X \in (\langle I \rangle^{(n+1)} \setminus \langle I \rangle^{(n)}) \exists X_1, X_2 \in \langle I \rangle^{(n)} \exists c_i \in \langle I \rangle[X_i, I] \cap \mathbf{C}_K \quad (i = 1, 2) \\ (X = X_1 \otimes X_2 \wedge c_1 \otimes c_2 \in \mathbf{C}_K \Rightarrow (c_1 \otimes c_2)r_I \in \langle I \rangle[X, I] \cap \mathbf{C}_K). \end{aligned}$$

b) One proves by induction over the complexity of Y :

$$\forall X \in \langle I \rangle \forall Y \in \langle I \rangle \exists c \in \langle I \rangle[X, Y] \quad (c \in \mathbf{C}_K).$$

The truth of the assertion for an arbitrary $X \in \langle I \rangle$ and for $Y \in \langle I \rangle^{(0)}$ was shown in a).

$$\begin{aligned} \forall X \in \langle I \rangle \forall n \in \mathbb{N} [\forall Y \in \langle I \rangle^{(n)} \exists c \in \langle I \rangle[X, Y] \quad (c \in \mathbf{C}_K) \Rightarrow \\ \Rightarrow \forall Y \in \langle I \rangle^{(n+1)} \exists c \in \langle I \rangle[X, Y] \quad (c \in \mathbf{C}_K)], \end{aligned}$$

since

$$\begin{aligned} \forall Y \in (\langle I \rangle^{(n+1)} \setminus \langle I \rangle^{(n)}) \exists Y_1, Y_2 \in \langle I \rangle^{(n)} \exists c_1 \in \langle I \rangle[X, Y_1] \cap \mathbf{C}_K \\ \exists c_2 \in \langle I \rangle[Y_2, Y] \cap \mathbf{C}_K \end{aligned}$$

$$(Y = Y_1 \otimes Y_2 \wedge c_1 \otimes c_2 \in \mathbf{C}_K \Rightarrow r_X^{-1}(c_1 \otimes c_2) \in \langle I \rangle[X, Y] \cap \mathbf{C}_K). \quad \blacksquare$$

Definitions 1.7. Let K^\bullet be a symmetric monoidal category in the sense of [3].

A sequence $(K^\bullet; d)$ is called *diagonal-symmetric monoidal category* (shortly *ds-category*) (in [2] considered in the strict case as a special Kronecker-category, in [13] as “diagonal-symmetrische Kategorie”), if $d = (d_A \in K[A, A \otimes A] \mid A \in |K|)$ is a family of morphisms of K such that

$$(D1) \quad \forall A, A' \in |K| \forall \varphi \in K[A, A'] \quad (\varphi d_{A'} = d_A(\varphi \otimes \varphi)),$$

$$(D2) \quad \forall A \in |K| \quad (d_A(d_A \otimes 1_A) = d_A(1_A \otimes d_A)a_{A,A,A}),$$

$$(D3) \quad \forall A \in |K| \quad (d_A s_{A,A} = d_A),$$

$$(D4) \quad \forall A, B \in |K| \quad ((d_A \otimes d_B)b_{A,A,B,B} = d_{A \otimes B})$$

are fulfilled.

(K^\bullet, d, t) is called *diagonal-terminal-symmetric monoidal category* (*dts-category*) ([2]), if (K^\bullet, d) is a *ds-category* with a family $t = (t_A \mid A \in |K|)$ of terminal morphisms $t_A \in K[A, I]$ such that the conditions

$$(T1) \quad \forall A, A' \in |K| \forall \varphi \in K[A, A'] (\varphi t_{A'} = t_A)$$

and

$$(DTR) \quad \forall A \in |K| (d_A(1_A \otimes t_A)r_A = 1_A)$$

are right.

$(K^\bullet; d, t, o)$ will be called *diagonal-halfterminal-symmetric monoidal category* or *Hoehnke category* (shortly *dhts-category*) ([5], [11], [13]), if d and t are morphism families as above and $o : I \rightarrow O$ is a distinguished morphism in K related to a distinguished object $O \in |K|$, such that

$$(D1) \quad \forall A, A' \in |K| \forall \varphi \in K[A, A'] (d_A(\varphi \otimes \varphi) = \varphi d_{A'}),$$

$$(DTR) \quad \forall A \in |K| (d_A(1_A \otimes t_A)r_A = 1_A),$$

$$(DTL) \quad \forall A \in |K| (d_A(t_A \otimes 1_A)l_A = 1_A),$$

$$(DTRL) \quad \forall A_1, A_2 \in |K| (d_{A_1 \otimes A_2}((1_{A_1} \otimes t_{A_2})r_{A_1} \otimes (t_{A_1} \otimes 1_{A_2})l_{A_2}) = 1_{A_1 \otimes A_2}),$$

$$(TT) \quad \forall A, B \in |K| (t_{A \otimes B} = (t_A \otimes t_B)t_{I \otimes I}),$$

$$(O1) \quad \forall A \in |K| (A \otimes O = O \otimes A = O),$$

$$(o1) \quad \forall A \in |K| \forall \varphi \in K[A, O] (t_A o = \varphi),$$

and

$$(o2) \quad \forall A \in |K| \forall \psi \in K[O, A] ((1_A \otimes t_O)r_A = \psi)$$

are fulfilled.

$(K^\bullet; d, t, \nabla, o)$ is called *diagonal-halfterminal-halfdiagonal-inversional-symmetric monoidal category* or *Hoehnke category with halfdiagonalinversions* (for short *dht ∇ s-category*, in [13] named *dht ∇ -symmetric category*), if $(K^\bullet; d, t, o)$ is a *dhts-category* endowed with a morphism family

$$\nabla = (\nabla_A \in K[A \otimes A, A] \mid A \in |K|) \text{ fulfilling}$$

$$(D_1^*) \quad \forall A \in |K| \quad (d_A \nabla_A = 1_A),$$

$$(D_2^*) \quad \forall A \in |K| \quad (\nabla_A d_A d_{A \otimes A} = d_{A \otimes A} (\nabla_A d_A \otimes 1_{A \otimes A})).$$

Any *ds*-, *dts*-, *dhts*-, and *dth ∇ s*-category, respectively, is called *strict*, if the underlying symmetric monoidal category is strictly monoidal.

The zero morphisms $o_{A,B}$ absorb all other morphisms at composition and \otimes -operation in any *dhts*-category, i.e.

$$(o3) \quad \forall A, A', B, B' \in |K| \quad \forall \rho \in K[A, A'], \quad \sigma \in K[B, B']$$

$$(\rho o_{A',B} = o_{A,B} \wedge o_{A,B} \sigma = o_{A,B'}),$$

$$(o4) \quad \forall A, B, C, D \in |K| \quad \forall \xi \in K[C, D]$$

$$(o_{A,B} \otimes \xi = o_{A \otimes C, B \otimes D} \wedge \xi \otimes o_{A,B} = o_{C \otimes A, B \otimes D}),$$

$$(o5) \quad \forall A \in |K| \quad (o_{O,A} = (1_A \otimes t_O) r_A = (t_O \otimes 1_A) l_A).$$

Because of (o1) and (o2), the unit morphism 1_O is identical with the zero morphism $o_{O,O}$.

The category Par of all partial functions between arbitrary sets is an example for a *dth ∇ s*-category.

In view of the properties of the category Par we will consider mainly *dhts*-categories fulfilling the conditions

$$(N_1) \quad \forall A, B \in |K| \quad (A \otimes B = O \Rightarrow (A = O \vee B = O)),$$

$$(N_2) \quad \forall A, B, C, D \in |K| \quad \forall \varphi \in K[A, B] \quad \forall \psi \in K[C, D]$$

$$(\varphi \otimes \psi = o_{A \otimes C, B \otimes D} \Rightarrow (\varphi = o_{A,B} \vee \psi = o_{C,D})).$$

$$(N_3) \quad I \neq O,$$

$$(N_4) \quad \forall A \in |K| \setminus \{\emptyset\} \quad (1_A \neq o_{A,A}).$$

Observe that $(K^\bullet; d)$ is a ds -category for each $dhts$ -category $(K^\bullet; d, t, o)$ and ∇ is the only family in a $dhth\nabla s$ -category with the properties (D_1^*) and (D_2^*) , cf. [11].

Any $dhts$ -category $\underline{K} = (K^\bullet; d, t, o)$ has the following properties:

- The class $T_K := \{\varphi \in K \mid \varphi t_{\text{cod}\varphi} = t_{\text{dom}\varphi}\}$ of so-called *total morphisms* of \underline{K} forms a dts -subcategory \underline{T}_K of \underline{K} ([12]).

- $(A \otimes B, (1_A \otimes t_B)r_A, (t_A \otimes 1_B)l_B)$

is a categorical product in \underline{T}_K , but not in the whole category \underline{K} . The morphisms

$$p_1^{A,B} := (1_A \otimes t_B)r_A \text{ and } p_2^{A,B} := (t_A \otimes 1_B)l_B$$

are called the *canonical projections* concerning A and B ([5]).

- The class Iso_K of all *isomorphisms* of K forms a symmetric monoidal subcategory Iso_K^\bullet and one has

$$Un_K \subseteq \mathbf{C}_K \subseteq Iso_K \subseteq Cor_K \subseteq T_K,$$

where Cor_K denotes the subcategory of all coretractions of K .

- The relation \leq defined by

$$\varphi \leq \psi := \Leftrightarrow \exists A, A' \in |K| (\varphi, \psi \in K[A, A'] \wedge \varphi = d_A(\varphi \otimes \psi)p_2^{A', A'})$$

is a partial order relation and it is compatible with composition and \otimes -operation of morphisms ([11]). Moreover, the following conditions are equivalent ([12]):

$$\varphi = d_A(\varphi \otimes \psi)p_2^{A', A'},$$

$$\varphi = d_A(\psi \otimes \varphi)p_1^{A', A'},$$

$$\varphi d_{A'} = d_A(\varphi \otimes \psi),$$

$$\varphi d_{A'} = d_A(\psi \otimes \varphi).$$

- Each morphism $\varphi \in K$ determines a so-called *subidentity* $\alpha(\varphi)$ as follows ([11]):

$$\alpha(\varphi) := d_{\text{dom}\varphi}(1_{\text{dom}\varphi} \otimes \varphi)p_1^{\text{dom}\varphi, \text{cod}\varphi} \leq 1_{\text{dom}\varphi}.$$

Moreover, each *dhth* ∇ *s*-category has the properties

$$(\text{h}\nabla_1) \quad \forall A, A' \in |K| \quad \forall \varphi \in K[A, A'] \quad (\nabla_{A'}\varphi d_{A'} = d_{A \otimes A}(\nabla_A\varphi \otimes (\varphi \otimes \varphi)\nabla_{A'}),$$

$$(\text{hT}_1) \quad \forall A, A' \in |K| \quad \forall \varphi \in K[A, A'] \quad (\varphi t_{A'} d_I = d_A(\varphi t_{A'} \otimes t_A)),$$

therefore $\nabla_A\varphi \leq (\varphi \otimes \varphi)\nabla_{A'}$ and $\varphi t_{A'} \leq t_A$ for all morphisms $\varphi \in K[A, A']$ and all objects $A, A' \in |K|$ ([15]).

Every morphism set $K[A, B]$ of a *dhth* ∇ *s*-category \underline{K} forms a meet-semilattice with respect to $\varphi \wedge \psi = d_A(\varphi \otimes \psi)\nabla_B$. This semilattice has the minimum $o_{A,B}$, maximal elements are the total morphisms. Especially, the morphism sets $K[A, I]$ possess a maximum, namely t_A .

The basic morphisms related to the distinguished object I in any symmetric monoidal category, any *dhts*-category, or even any *dhth* ∇ *s*-category have some interesting properties as follows:

Lemma 1.8. *Let K^\bullet be a symmetric monoidal category. Then one has*

$$a_{I,I,I} = r_I^{-1} \otimes r_I.$$

Moreover, every *dhts*-category \underline{K} has in addition the properties

$$d_I = r_I^{-1}, \quad r_I d_I = 1_{I \otimes I}, \quad t_I = 1_I \quad ([11]), \quad t_{I \otimes I} = r_I,$$

$$i \in \text{Iso}_K[I, I] \Rightarrow i = t_I,$$

$$\forall X \in |K| \quad \forall x \in K[I, X] \quad (x \in \text{Iso}_K \Rightarrow x^{-1} = t_X).$$

Finally, if \underline{K} is a *dhth* ∇ *s*-category, then the additional property

$$\nabla_I = r_I$$

is true.

Proof. The identity $a_{A,I,B}(r_A \otimes 1_B) = 1_A \otimes l_B$ is one of the defining properties of monoidal-symmetric categories, hence $a_{I,I,I}(r_I \otimes 1_I) = 1_I \otimes r_I$ by $r_I = l_I$ and $a_{I,I,I} = (r_I^{-1} \otimes r_I)$, since all right-identity morphisms are isomorphisms.

In any *dhts*-category one has the defining identity $d_A(1_A \otimes t_A)r_A = 1_A$, hence $1_I = d_I(1_I \otimes t_I)r_I = d_I(1_I \otimes 1_I)r_I = d_I r_I$, since $t_I = 1_I$, consequently $d_I = r_I^{-1}$ and $r_I d_I = 1_{I \otimes I}$.

Each coretraction $\varphi \in K[A, B]$ of a *dhts*-category has the property $\varphi t_B = t_A$. Because d_I is even an isomorphism, one observes $d_I t_{I \otimes I} = t_I = 1_I$, therefore $t_{I \otimes I} = 1_{I \otimes I} t_{I \otimes I} = r_I d_I t_{I \otimes I} = r_I 1_I = r_I$.

One of the characterizing conditions of the diagonal inversions in a *dhth* ∇ *s*-category is $d_A \nabla_A = 1_A$. Therefore, $\nabla_I = 1_{I \otimes I} \nabla_I = r_I d_I \nabla_I = r_I$ as above. Now let $i \in K[I, I]$ be an isomorphism of a *dhts*-category \underline{K} . Then $i = i 1_I = i t_I = t_I$, because of $1_i = t_I$.

Let $x \in K[I, X]$ be an isomorphism in a *dhts*-category \underline{K} . Then one obtains in the same manner as above $1_I = t_I = x t_X$, hence the assertion. ■

Remark 1.9. Let \underline{K} be a *dhts*-category. Then its object class $|K|$ forms an illegitimate algebra $(|K|, \otimes, I, O)$ of type $(2, 0, 0)$. Let J be a nonempty set such that $J \cap \{I, O\} = \emptyset$. Then J generates in $|K|$ a subalgebra H° of type $(2, 0, 0)$:

$$H^{\circ(0)} := J \cup \{I, O\}, \quad H^{\circ(n+1)} := H^{\circ(n)} \cup \{X \otimes Y \mid X, Y \in H^{\circ(n)}\},$$

$$H^\circ := \bigcup_{n \in \mathbb{N}} H^{\circ(n)}.$$

The *dhts*-subcategory of \underline{K} generated by H° , respectively by J , will be denoted by \underline{H}° . Obviously, \underline{H}° is again a small category.

Let \underline{K} be a strict *dhts*-category. Then the algebra $S^\circ := (H^\circ, \otimes, I, O)$ generated by a set J is a monoid with unit I and zero O .

2. HOEHNKE THEORIES

Let \mathcal{G} denote the variety of all algebras of type type $(2, 0)$ (groupoids with a distinguished element I). Note that the distinguished element I does not play the role of a unit element in general. By the principles of General Algebra, every set J determines in \mathcal{G} a free \mathcal{G} -algebra $\mathbf{F}_{\mathcal{G}}(J)$ freely generated

by J . The algebra $\mathbf{F}_{\mathcal{G}}(J)$ contains a subalgebra $\langle I \rangle$ consisting of all possible products of I as follows:

$$\langle I \rangle^{(0)} := \{I\}, \langle I \rangle^{(n+1)} := \langle I \rangle^{(n)} \cup \{X \otimes Y \mid X, Y \in \langle I \rangle^{(n)}\}, \langle I \rangle := \bigcup_{k \in \mathbb{N}} \langle I \rangle^{(k)}.$$

Every algebra $\underline{A} = (A; \otimes, I) \in \mathcal{G}$ can be transferred into an algebra $(A; \otimes, I, O)$ of type $(2, 0, 0)$ by addition of a distinguished element O with the property $\forall X \in A (X \otimes O = O = O \otimes X)$.

By \mathcal{G}° shall be denoted the variety of all algebras $(A; \otimes, I, O)$ of type $(2, 0, 0)$ (groupoids with distinguished element I and zero element O) such that $\forall X \in A (X \otimes O = O = O \otimes X)$. $\mathbf{F}_{\mathcal{G}^\circ}(J)$ denotes the free \mathcal{G}° -algebra freely generated by a set J such that $J \cap \{I, O\} = \emptyset$. Clearly, $\mathbf{F}_{\mathcal{G}^\circ}(J)$ contains the trivial subalgebra $\langle I \rangle^\circ$ with the carrier set $\langle I \rangle^\circ = \langle I \rangle \cup \{O\}$.

Let \mathcal{M} be the variety of all monoids (algebras of type $(2, 0)$) and let \mathcal{M}° be the variety of all monoids with absorbing zero (algebras of type $(2, 0, 0)$ too).

The free \mathcal{M} -algebra (\mathcal{M}° -algebra) freely generated by J will be denoted by $\mathbf{F}_{\mathcal{M}}(J)$ ($\mathbf{F}_{\mathcal{M}^\circ}(J)$). The trivial subalgebra $\langle I \rangle$ ($\langle I \rangle^\circ$) has the carrier set $\langle I \rangle = \{I\}$ ($\langle I \rangle^\circ = \{I, O\}$).

The identical embedding functions from J into the corresponding algebras will be denoted as follows:

$$\begin{aligned} \iota_H : J &\hookrightarrow \mathbf{F}_{\mathcal{G}}(J), & \iota_{H^\circ} : J &\hookrightarrow \mathbf{F}_{\mathcal{G}^\circ}(J), \\ \iota_S : J &\hookrightarrow \mathbf{F}_{\mathcal{M}}(J), & \iota_{S^\circ} : J &\hookrightarrow \mathbf{F}_{\mathcal{M}^\circ}(J). \end{aligned}$$

Definition 2.1 ([5]). Let \mathbf{T} be a *dhts*-category, a *dhth* ∇ *s*-category, or a *dts*-category and let J be a nonempty set of objects of \mathbf{T} such that $I, O \notin J$.

Then \mathbf{T} will be called

- J*-sorted *dhts*-theory or *J*-sorted *Hoehnke theory*,
- J*-sorted *dhth* ∇ *s*-theorie or
- J*-sorted *Hoehnke theory with halfdiagonalinversions*,
- J*-sorted *dts*-theory, respectively,

if $(|\mathbf{T}|; \otimes, I, O)$ is a free \mathcal{G}° -algebra freely generated by J ($(|\mathbf{T}|; \otimes, I)$ is a free \mathcal{G} -algebra freely generated by J , $I \notin J$).

The class of all *J*-sorted *dhts*-theories (*J*-sorted *dhth* ∇ *s*-theories, *J*-sorted *dts*-theories) will be denoted by $|Th_{dht}^\circ(J)|$ ($|Th_{dhth\nabla}^\circ(J)|$, $|Th_{dt}(J)|$).

Besides the theory concept above we consider the following, more artificial, but simpler one, which arises in strict monoidal categories by replacing of the groupoid $\mathbf{F}_{\mathcal{G}^\circ}(J)$ ($\mathbf{F}_{\mathcal{G}}(J)$) by the monoid $\mathbf{F}_{\mathcal{M}^\circ}(J)$ ($\mathbf{F}_{\mathcal{M}}(J)$). So, one defines

Definition 2.2. Let \mathbf{T} be a *dhts*-category, a *dhth* ∇ *s*-category, or a *dts*-category such that the underlying symmetric monoidal category \mathbf{T}^\bullet is strictly monoidal, i.e. all the morphisms a , r , and l are unit-morphisms only ($A \otimes (B \otimes C) = (A \otimes B) \otimes C$, $A \otimes I = A = I \otimes A$, $a_{A,B,C} = 1_{A \otimes B \otimes C}$, $r_A = 1_A = l_A$ for all $A, B, C \in |\mathbf{T}|$).

Then \mathbf{T} will be called

J-sorted strict *dhts*-theory or strict *J*-sorted Hoehnke theory,
J-sorted strict *dhth* ∇ *s*-theory or
 strict *J*-sorted Hoehnke theory with halfdiagonalinversions,
J-sorted strict *dts*-theory, respectively,

if there exists a nonempty set J in $|\mathbf{T}|$ such that $I, O \notin J$ and $(|\mathbf{T}|; \otimes, I, O)$ is a free \mathcal{M}° -algebra ($(|\mathbf{T}|; \otimes, I)$ is a free \mathcal{M} -algebra) freely generated by J . The class of all *J*-sorted strict *dhts*-theories (*J*-sorted strict *dhth* ∇ *s*-theories, *J*-sorted strict *dts*-theories) will be denoted by

$$|sTh_{dht}^\circ(J)| \quad (|sTh_{dhth\nabla}^\circ(J)|, |sTh_{dt}(J)|).$$

The categories of the classes $|Th_{dht}^\circ(J)|$, $|Th_{dhth\nabla}^\circ(J)|$, $|sTh_{dht}^\circ(J)|$, and $|sTh_{dhth\nabla}^\circ(J)|$ shortly will called *partial theories* (*Hoehnke theories*) and categories of $|Th_{dt}(J)|$ and $|sTh_{dt}(J)|$ are named *total theories*.

For a given set J one has on the one hand the free algebra $\mathbf{F}_{\mathcal{G}^\circ}(J)$ and on the other hand the free algebra $\mathbf{F}_{\mathcal{M}^\circ}(J)$ and both are algebras of the variety \mathcal{G}° of type $(2, 0, 0)$. Therefore, there arises the question about a connection between the two algebras.

Lemma 2.3. Let $\mathbf{F}_{\mathcal{G}^\circ}(J) =: (H^\circ; \otimes, I, O)$, $\mathbf{F}_{\mathcal{M}^\circ}(J) =: (S^\circ; \otimes, I, O)$, $\mathbf{F}_{\mathcal{G}}(J) =: (H; \otimes, I)$, and $\mathbf{F}_{\mathcal{M}}(J) =: (S; \otimes, I)$ be the algebras defined as above. Then there is exactly one homomorphism $W^* : \mathbf{F}_{\mathcal{G}^\circ}(J) \rightarrow \mathbf{F}_{\mathcal{M}^\circ}(J)$ ($W^* : \mathbf{F}_{\mathcal{G}}(J) \rightarrow \mathbf{F}_{\mathcal{M}}(J)$) such that $\iota_{H^\circ} W^* = \iota_{S^\circ}$ ($\iota_H W^* = \iota_S$).

The mapping W^* works as follows:

$$I \mapsto I =: IW^*, \quad O \mapsto O =: OW^*, \quad J \ni A \mapsto A =: AW^*, \\ \forall X, Y \in H \quad ((X \otimes Y)W^* = XW^* \otimes YW^*).$$

Proof. Let $\underline{\mathbf{T}} \in |\mathit{Th}_{dht}^\circ(J)|$. The algebra $\mathbf{F}_{\mathcal{M}^\circ}(J) = (S^\circ; \otimes, I, O)$, generated by J , belongs to \mathcal{G}° . Since $(H^\circ := |\mathbf{T}|; \otimes, I, O)$ is a free \mathcal{G}° -algebra freely generated by J , there is exactly one homomorphism W^* such that $\iota_{H^\circ} W^* = \iota_{S^\circ}$ and this homomorphism is surjective. The assertion about the working of the mapping becomes clear since ι_{S° is the identical embedding of J into S° .

The statement concerning groupoids and monoids without zero will be proved in the same manner. \blacksquare

Corollary 2.4. *The mapping $W^* : H^\circ \rightarrow S^\circ$ has the following properties:*

$$\forall X \in \langle I \rangle \quad (XW^* = I),$$

$$\forall Y \in H^\circ \quad \forall X \in \langle I \rangle \quad ((Y \otimes X)W^* = (X \otimes Y)W^* = YW^*),$$

$$\forall X, Y, Z \in H^\circ \quad ((X \otimes (Y \otimes Z))W^* = ((X \otimes Y) \otimes Z)W^*),$$

$$\forall X \in H^\circ \setminus \langle I \rangle^\circ \quad \exists!! A_1, A_2, \dots, A_n \quad (XW^* = A_1 \otimes A_2 \otimes \dots \otimes A_n).$$

Proof. The first assertion one proves by induction over the complexity of the elements of $\langle I \rangle$.

By Lemma 2.3, $IW^* = I$. Assume that for any $n \in \mathbb{N}$ the condition

$$\forall Y \in \langle I \rangle^{(n)} \quad (YW^* = I)$$

is valid. Then

$$\forall X \in \langle I \rangle^{(n+1)} \setminus \langle I \rangle^{(n)} \quad \exists X_1, X_2 \in \langle I \rangle^{(n)}$$

$$(XW^* = (X_1 \otimes X_2)W^* = X_1W^* \otimes X_2W^* = I \otimes I = I),$$

hence $\forall n \in \mathbb{N} \quad \forall X \in \langle I \rangle^{(n)} \quad (XW^* = I)$.

Because of $(X \otimes Y)W^* = XW^* \otimes YW^*$, $XW^* = I$ for every $X \in \langle I \rangle$ and I is the unit element in the monoid, the second claim becomes true.

Let X, Y , and Z be elements of $|T| = H^\circ$. Then XW^* , YW^* , and ZW^* are elements of the monoid \underline{S}° and

$$(X \otimes (Y \otimes Z))W^* = XW^* \otimes YW^* \otimes ZW^* = ((X \otimes Y) \otimes Z)W^*.$$

Because of

$$H = \bigcup_{k \in \mathbb{N}} H^{(k)}, \quad H^{(0)} := J \cup \{I\},$$

$$H^{(n+1)} := H^{(n)} \cup \{X \otimes Y \mid X, Y \in H^{(n)}\}, \quad n \in \mathbb{N},$$

one shows the existence of such a representation by induction over the complexity of X .

$$X \in H^{(0)} \setminus \langle I \rangle \Rightarrow X = A \in J \Rightarrow XW^* = AW^* = A.$$

Assuming that for any $n \in \mathbb{N}$ each $X \in H^{(n)} \setminus \langle I \rangle$ fulfills the assertion one investigates an arbitrary $Y \in H^{(n+1)} \setminus H^{(n)} \setminus \langle I \rangle$. Then there are $X_1, X_2 \in H^{(n)} \setminus \langle I \rangle$ such that $YW^* = (X_1 \otimes X_2)W^* = X_1W^* \otimes X_2W^*$, hence there are $A_1, \dots, A_j, B_1, \dots, B_k \in J$ such that $YW^* = (X_1 \otimes X_2)W^* = X_1W^* \otimes X_2W^* = A_1 \otimes \dots \otimes A_j \otimes B_1 \otimes \dots \otimes B_k$.

The uniqueness of the factors of a \otimes -product which are elements of J is a consequence of the fact that $(S^\circ; \otimes, I, O)$ is a free \mathcal{M}° -algebra freely generated by J . ■

Lemma 2.5. *Let be given H° and S° as above related to a fixed set J . Then there is a function $W : S^\circ \rightarrow H^\circ$ such that*

$$(W1) \quad WW^* = 1_{S^\circ} \text{ and}$$

$$(W2) \quad \forall A, B \in S^\circ \quad (A \otimes B = (AW \otimes BW)W^*).$$

The function $\Phi : H^\circ \rightarrow H^\circ$ defined by $\Phi := W^*W$ has the properties

$$(W3) \quad \forall X \in \langle I \rangle \quad (X\Phi = I),$$

$$(W4) \quad \forall X \in H \setminus \langle I \rangle \quad \exists!! \ A_1, \dots, A_n \in J \quad \left(X\Phi = \bigotimes_{j=1}^n A_j \right),$$

$$(W5) \quad \forall X_1, X_2, Y_1, Y_2 \in H^\circ$$

$$((X_1 \otimes X_2)\Phi = (Y_1 \otimes Y_2)\Phi \Leftrightarrow (X_1\Phi) \otimes (X_2\Phi) = (Y_1\Phi) \otimes (Y_2\Phi)).$$

Proof. Ad (W1): Defining

$$OW := O, IW := I, \forall A_1, \dots, A_n \in J \left((A_1 \otimes \dots \otimes A_n)W := \bigotimes_{j=1}^n A_j \right)$$

one gets immediately $WW^* = 1_{S^0}$.

Ad (W2): The assertion is trivial for $A = O$ or $B = O$. The same is true if $A = I$ or $B = I$. Now let $A, B \in S \setminus \{I\}$. Then, by definition,

$$\begin{aligned} A \otimes B &= A_1 \otimes \dots \otimes A_n \otimes B_1 \otimes \dots \otimes B_m = \left(\bigotimes_{k=1}^n A_k \right) W^* \otimes \left(\bigotimes_{j=1}^m B_j \right) W^* \\ &= (AW)W^* \otimes (BW)W^* = (AW \otimes BW)W^*. \end{aligned}$$

Ad (W3): The condition is valid for $X \in \{I, O\}$, since

$$I\Phi = IW^*W = IW = I \text{ and } O\Phi = OW^*W = OW = O.$$

Let X be an arbitrary element of $\langle I \rangle$. Then

$$X\Phi = (XW^*)W = IW = I.$$

Ad (W4): For all $X \in H \setminus \langle I \rangle$ one has

$$X\Phi = (XW^*)W = (A_1 \otimes \dots \otimes A_n)W = \bigotimes_{j=1}^n A_j$$

and, by the properties of a free algebra,

$$\bigotimes_{j=1}^n A_j = \bigotimes_{k=1}^m A'_k \Rightarrow n = m \wedge A_j = A'_j \text{ for all } j \in \{1, \dots, n\}.$$

Ad (W5):

$$(X_1 \otimes X_2)\Phi = (Y_1 \otimes Y_2)\Phi \Leftrightarrow (X_1 \otimes X_2)W^*W = (Y_1 \otimes Y_2)W^*W$$

$$\Leftrightarrow (X_1 \otimes X_2)W^* = (Y_1 \otimes Y_2)W^*$$

$$\Leftrightarrow X_1W^* \otimes X_2W^* = Y_1W^* \otimes Y_2W^*$$

$$\Leftrightarrow X_1W^* = Y_1W^* \wedge X_2W^* = Y_2W^*$$

(\underline{S}° is a free algebra)

$$\Leftrightarrow X_1W^*W = Y_1W^*W \wedge X_2W^*W = Y_2W^*W$$

$$\Leftrightarrow X_1\Phi = Y_1\Phi \wedge X_2\Phi = Y_2\Phi$$

$$\Leftrightarrow X_1\Phi \otimes X_2\Phi = Y_1\Phi \otimes Y_2\Phi \text{ } (\underline{H}^\circ \text{ is a free algebra}).$$

■

Observe that the function $\Phi : H^\circ \rightarrow H^\circ$ maps O onto O , all elements of $\langle I \rangle \subseteq H$ onto I , and all elements $X \in H \setminus \langle I \rangle$ onto an \otimes -product of elements of J in canonical brackets consisting exactly of the factors of X which are different from I in the same order.

Lemma 2.6. *Let be \underline{H}° , \underline{S}° , $\Phi : H^\circ \rightarrow H^\circ$ as above. Then*

$$\forall X, Y, Z \in H^\circ \quad ((X \otimes (Y \otimes Z))\Phi = ((X \otimes Y) \otimes Z)\Phi),$$

$$\forall n \in \mathbb{N} \setminus \{0\} \quad \forall A_1, \dots, A_n \in J \quad \left(\left(\left(\begin{array}{c} n \\ \otimes \\ j=1 \end{array} A_j \right) \Phi = \begin{array}{c} n \\ \otimes \\ j=1 \end{array} A_j \right), \right.$$

$$\left. \forall X \in \langle I \rangle \quad \forall Y \in H^\circ \quad ((Y \otimes X)\Phi = (X \otimes Y)\Phi = Y\Phi). \right.$$

Proof.

$$(X \otimes (Y \otimes Z))\Phi = (X \otimes (Y \otimes Z))W^*W = (XW^* \otimes (YW^* \otimes ZW^*))W$$

$$= ((XW^* \otimes YW^*) \otimes ZW^*)W = ((X \otimes Y) \otimes Z)\Phi.$$

$$\left(\begin{array}{c} n \\ \otimes \\ j=1 \end{array} A_j \right) \Phi = \left(\begin{array}{c} n \\ \otimes \\ j=1 \end{array} A_j \right) W^*W = \left(\begin{array}{c} n \\ \otimes \\ j=1 \end{array} A_j \right) W = \begin{array}{c} n \\ \otimes \\ j=1 \end{array} A_j.$$

$$(Y \otimes X)\Phi = (Y \otimes X)W^*W = (YW^* \otimes XW^*)W = (YW^* \otimes I)W = YW^*W = Y\Phi,$$

$$(X \otimes Y)\Phi = (X \otimes Y)W^*W = (XW^* \otimes YW^*)W = (I \otimes YW^*)W = YW^*W = Y\Phi. \quad \blacksquare$$

Corollary 2.7. *Let \mathbf{T} be any J -sorted Hoehnke theory and let $\Phi : H^\circ \rightarrow H^\circ$ be defined as above. Then there is exactly one central morphism $c_X := c_{X, X\Phi}$ in $\mathbf{C}_{\mathbf{T}}$ for every $X \in |\mathbf{T}|$. The same statement is true, if \mathbf{T} is a J -sorted *dts*-theory and $\Phi : H \rightarrow H$.*

Moreover, $\forall X, Y \in |\mathbf{T}| (X\Phi = Y\Phi \Rightarrow \exists c_{X,Y} \in \mathbf{C}_{\mathbf{T}}[X, Y])$.

Proof. The proof is organized by induction over the complexity of the objects $X \in |\mathbf{T}| = H^\circ$.

Because of $X\Phi = X$ for every $X \in J \cup \{I, O\} = H^{\circ(0)}$, $1_X \in \mathbf{C}_{\mathbf{T}}[X, X\Phi]$, hence the start of induction is verified.

Let c_X exist in $\mathbf{C}_{\mathbf{T}}$ for any $X \in H^{\circ(n)}$ and an arbitrary $n \in \mathbb{N}$. Let be $X \in H^{\circ(n+1)} \setminus H^{\circ(n)}$. Then there are $X_1, X_2 \in H^{\circ(n)}$ such that $X = X_1 \otimes X_2$ and $c_{X_1} \in \mathbf{C}_{\mathbf{T}}[X_1, X_1\Phi]$, $c_{X_2} \in \mathbf{C}_{\mathbf{T}}[X_2, X_2\Phi]$, hence $(c_{X_1} \otimes c_{X_2}) \in \mathbf{C}_{\mathbf{T}}[X, X_1\Phi \otimes X_2\Phi]$.

Since $X_1\Phi = \bigotimes_{j=1}^n A_j$ and $X_2\Phi = \bigotimes_{j=n+1}^{n+m} A_j$ for suitable $A_j \in J$, $1 \leq j \leq n+m$, there is the canonical associativity isomorphism

$$a^{(n,m)} \langle X_1\Phi, X_2\Phi \rangle : X_1\Phi \otimes X_2\Phi \rightarrow (X_1 \otimes X_2)\Phi = X\Phi \text{ in } \mathbf{C}_{\mathbf{T}},$$

therefore,

$$c_X := (c_{X_1} \otimes c_{X_2})a^{(n,m)} \langle X_1\Phi, X_2\Phi \rangle \in \mathbf{C}_{\mathbf{T}}[X, X\Phi].$$

So, the existence of a central morphism c_X for every $X \in |\mathbf{T}| = H^\circ$ is proved. Moreover, $X\Phi = Y\Phi$ is sufficient for $c_{X,Y} := c_X(c_Y)^{-1} \in \mathbf{C}_{\mathbf{T}}[X, Y]$.

The uniqueness is again a consequence of the coherence principle.

The claim concerning the *dts*-case will be proved similarly. \blacksquare

The function Φ defined as above induces in a natural manner a functor from a J -sorted theory \mathbf{T} into itself with additional interesting properties. This properties concern the monoidal structure of \mathbf{T} .

3. STRUCTURE PRESERVING FUNCTORS

Considering different symmetric monoidal categories K^\bullet and K'^\bullet one has to distinguish between the operations and the basic morphisms of K^\bullet and those of K'^\bullet , respectively, for instance between $r_A^{(K)}$ and $r_X^{(K')}$. If there is not danger of confusion, the upper index will be omitted.

Definition 3.1 ([14]). A functor $F : K^\bullet \rightarrow K'^\bullet$ between symmetric monoidal categories K^\bullet and K'^\bullet is called *monoidal*, iff there exists in K' a family of morphisms

$$\tilde{F} = (\tilde{F}\langle X, Y \rangle : XF \otimes YF \rightarrow (X \otimes Y)F \mid X, Y \in |K|)$$

and a morphism

$$i_F : I' \rightarrow IF,$$

such that the following conditions are fulfilled:

- (F \sim) $\forall X, Y \in |K| \ (\tilde{F}\langle X, Y \rangle \in Iso_{K'})$,
- (FI) $i_F \in Iso_{K'}$,
- (FA) $\forall X, Y, Z \in |K| \left(\left(1_{XF}^{(K')} \otimes \tilde{F}\langle Y, Z \rangle \right) \tilde{F}\langle X, Y \otimes Z \rangle \left(a_{X, Y, Z}^{(K)} F \right) \right. \\ \left. = a_{XF, YF, ZF}^{(K')} \left(\tilde{F}\langle X, Y \rangle \otimes 1_{ZF}^{(K')} \right) \tilde{F}\langle X \otimes Y, Z \rangle \right)$,
- (FR) $\forall X \in |K| \ \left(\tilde{F}\langle X, I \rangle \left(r_X^{(K)} F \right) = \left(1_{XF}^{(K')} \otimes i_F^{-1} \right) r_{XF}^{(K')} \right)$,
- (FS) $\forall X, Y \in |K| \ \left(\tilde{F}\langle X, Y \rangle \left(s_{X, Y}^{(K)} F \right) = s_{XF, YF}^{(K')} \tilde{F}\langle Y, X \rangle \right)$,
- (FM) $\forall \varphi : X \rightarrow Y, \psi : U \rightarrow V \in K \ \left((\varphi F \otimes \psi F) \tilde{F}\langle Y, V \rangle = \right. \\ \left. = \tilde{F}\langle X, U \rangle (\varphi \otimes \psi) F \right)$.

A monoidal functor $F : K^\bullet \rightarrow K'^\bullet$ is called *strictly monoidal*, iff all morphisms of the family \tilde{F} as well as the morphism i_F are unit morphisms only.

Corollary 3.2 ([14]). *Let $F : K^\bullet \rightarrow K'^\bullet$ be a monoidal functor between symmetric monoidal categories with reference to \tilde{F}, i_F . Then*

$$(FL) \ \forall X \in |K| \ \left(\tilde{F}\langle I, X \rangle \left(l_X^{(K)} F \right) = \left(i_F^{-1} \otimes 1_{XF}^{(K')} \right) l_{XF}^{(K')} \right). \quad \blacksquare$$

In applications to theories of algebraic structures, functors $F : \underline{K} \rightarrow \underline{K}'$ between *dhts*-categories, *dht ∇ s*-categories, or *dts*-categories are of interest which preserve in addition to the functor properties the *dhts*-, *dht ∇ s*-, and the *dts*-structure, respectively, with respect to a family $\tilde{F} = (\tilde{F}\langle X, Y \rangle \mid X, Y \in |K|)$ of isomorphisms $\tilde{F}\langle X, Y \rangle : XF \otimes YF \rightarrow (X \otimes Y)F$ in \underline{K}' and an isomorphism i_F between I' and IF , where I and I' are the distinguished objects in \underline{K} and \underline{K}' , respectively, ([5], [12], [14]). All symmetric monoidal categories with additional structures mentioned above are *ds*-categories. Of importance is the fact that a monoidal functor between at least *ds*-categories, which respects the diagonal morphisms except for isomorphisms, respects the canonical partial order relation and the distinguished terminal morphisms and the distinguished diagonal inversions, respectively, except for isomorphisms.

Definition 3.3 ([14]). A monoidal functor $F : \underline{K} \rightarrow \underline{K}'$ between *ds*-categories \underline{K} and \underline{K}' is called *d-monoidal*, if in addition the condition

$$(FD) \quad \forall A \in |K| \quad \left(d_A^{(K)} F = d_{AF}^{(K')} \tilde{F}\langle A, A \rangle \right)$$

holds with reference to the corresponding isomorphisms \tilde{F} and i_F . A strictly monoidal functor F fulfilling the condition (FD) is called *strictly d-monoidal*.

Obviously, the identical functor $\mathbf{1}_K$ of K^\bullet forms a strictly monoidal functor with respect to

$$\tilde{\mathbf{1}}_K = (\tilde{\mathbf{1}}_K\langle X, Y \rangle = 1_{XF \otimes YF} \mid X, Y \in |K|), i_{\mathbf{1}_K} = 1_I$$

and the constant functor $E : K^\bullet \rightarrow K'^\bullet (X \mapsto I', \varphi \mapsto 1'_{I'})$ with reference to

$$\tilde{E} = (\tilde{E}\langle X, Y \rangle = 1'_{I'} \mid X, Y \in |K|), i_E = 1'_{I'},$$

too, where K^\bullet and K'^\bullet are arbitrary symmetric monoidal categories.

Both functors are even *d*-monoidal functors, if $\underline{K} = (K^\bullet; d)$ and $\underline{K}' = (K'^\bullet; d')$ are *ds*-categories.

Moreover: Each *d*-monoidal functor $F : \underline{K} \rightarrow \underline{K}'$ between *dhts*-categories possesses the following properties with respect to the corresponding \tilde{F} , i_F ([11], [14]):

$$(FI^*) \quad t_{IF}^{(K')} = i_F^{-1},$$

$$(Fmon) \quad \forall \varphi, \psi \in K \quad (\varphi \leq \psi \Rightarrow \varphi F \leq \psi F),$$

$$(FT) \quad \forall X \in |K| \quad \left(t_X^{(K)} F t_{IF}^{(K')} = t_{XF}^{(K')} \right),$$

$$(FP) \quad \forall X, Y \in |K| \quad \left(p_j^{(K)X,Y} F = (\tilde{F}\langle X, Y \rangle)^{-1} p_j^{(K')X,YF} ; \quad j = 1, 2 \right),$$

$$(FE) \quad \forall A \in |K| \quad \left(e \leq 1_A^{(K)} \Rightarrow eF \leq 1_{AF}^{(K')} \right),$$

$$(FE\alpha) \quad \forall X, Y \in |K| \quad \forall \varphi \in K[X, Y] \quad \left((\alpha^{(K)}(\varphi))F = \alpha^{(K')}(\varphi F) \right).$$

Let $\underline{K}, \underline{K}'$ be *d*hth ∇ s-categories and let $F : \underline{K} \rightarrow \underline{K}'$ be a *d*-monoidal functor. Then, in addition to the the properties above, the following holds ([14]):

$$(Finf) \quad \forall X, Y \in |K| \quad \forall \varphi, \psi \in K[X, Y] \quad \left(\left(d_X^{(K)}(\varphi \otimes \psi) \nabla_Y^{(K)} \right) \right. \\ \left. F = d_{XF}^{(K')}(\varphi F \otimes \psi F) \nabla_{YF}^{(K')} \right),$$

$$(Finj) \quad \forall X, Y \in |K| \quad \forall \varphi \in K[X, Y] \quad \left((\varphi \otimes \varphi) \nabla_Y^{(K)} = \nabla_X^{(K)} \varphi \right. \\ \left. \Rightarrow (\varphi F \otimes \varphi F) \nabla_{YF}^{(K')} = \nabla_{XF}^{(K')}(\varphi F) \right),$$

$$(F\nabla) \quad \forall X \in |K| \quad \left(\nabla_{XF}^{(K')} = \tilde{F}\langle X, X \rangle \nabla_X^{(K)} F \right),$$

$$(F\nabla_1) \quad \forall X, Y, U \in |K| \quad \forall \varphi \in K[X, U] \quad \forall \psi \in K[Y, U] \quad \left(((\varphi \otimes \psi) \nabla_U^{(K)}) F \right. \\ \left. = \tilde{F}\langle X, Y \rangle \left((\varphi \otimes \psi) F \right) \nabla_{UF}^{(K')} \right),$$

$$(F\nabla_2) \quad \forall X, Y \in |K| \quad \forall \varphi, \psi \in K[X, Y] \quad \left(((\varphi \otimes \psi) \nabla_Y^{(K)}) = \nabla_X^{(K)} \varphi \right. \\ \left. \Rightarrow (\varphi F \otimes \psi F) \nabla_{YF}^{(K')} = \nabla_{XF}^{(K')}(\varphi F) \right).$$

Obviously, property (Finj) is a special case of (F ∇_2) and it expresses once more the monotony of the functor F , namely $\varphi \leq \psi \Rightarrow \varphi F \leq \psi F$.

The so-called zero functor $Z : \underline{K} \rightarrow \underline{K}'$ is defined by $XZ = O^{(K')}$ for all objects $X \in |K|$ and $\varphi Z = 1_{O^{(K')}}^{(K')}$ for all morphisms $\varphi \in K$. Trivially, this functor is a *d*-monoidal one.

Proposition 3.4 ([14]). *Let $F : \underline{K} \rightarrow \underline{K}'$ be a d -monoidal functor between Hoehnke categories such that $F \neq Z$. Then one obtains:*

$$\forall X \in |K| \quad \left(\tilde{F}\langle X, O \rangle = \tilde{F}\langle O, X \rangle = 1_{O^{(K')}} \right),$$

$$\forall X, Y \in |K| \quad \left(o_{X,Y}^{(K)} F = o_{XF,YF}^{(K')} \right),$$

$$o^{(K)} F = t_{IF}^{(K')} o^{(K')} \quad \left(\Leftrightarrow o^{(K')} = i_F(o^{(K)} F) \right). \quad \blacksquare$$

By the structure of any Hoehnke categories \underline{K} and \underline{K}' , each functor $F : \underline{K} \rightarrow \underline{K}'$ determines with respect to every pair of objects $X, Y \in |K|$ the morphism

$$F^*\langle X, Y \rangle := d_{(X \otimes Y)F}^{(K')} \left(p_1^{(K)X,Y} F \otimes p_2^{(K)X,Y} F \right) \in K'[(X \otimes Y)F, XF \otimes YF]$$

in the category K' .

Proposition 3.5 ([5]). *In the case that $F : \underline{K} \rightarrow \underline{K}'$ is a d -monoidal functor with reference to \tilde{F} and i_F , the morphisms $\tilde{F}\langle X, Y \rangle$ are uniquely determined by*

$$(\tilde{F}\langle X, Y \rangle)^{-1} = d_{(X \otimes Y)F}^{(K')} \left(p_1^{(K)X,Y} F \otimes p_2^{(K)X,Y} F \right) = F^*\langle X, Y \rangle. \quad \blacksquare$$

Moreover:

Theorem 3.6 ([5], [14]). *Assume that $F : \underline{K} \rightarrow \underline{K}'$ is any functor from a $dhts$ -category \underline{K} into a $dhts$ -category \underline{K}' satisfying the following conditions:*

$$(F^*) \quad \forall X, Y \in |K| (F^*\langle X, Y \rangle \in Iso_{K'}),$$

$$(FI^*) \quad t_{IF}^{(K')} \in Iso_{K'},$$

$$(FM^*) \quad \forall \varphi, \psi \in K \quad ((\varphi \otimes \psi)F F^*\langle X', Y' \rangle = F^*\langle X, Y \rangle(\varphi F \otimes \psi F)).$$

Then $F : \underline{K} \rightarrow \underline{K}'$ is d -monoidal with reference to the morphisms

$$\tilde{F}\langle X, Y \rangle := (F^*\langle X, Y \rangle)^{-1}, \quad i_F := t_{IF}^{(K')^{-1}}. \quad \blacksquare$$

The statements in 3.5 and 3.6 allow us to speak about d -monoidal functors between Hoehnke categories as such.

Hoehnke has shown in [5] that the composition of dht -symmetric functors $F : \underline{K} \rightarrow \underline{K}'$ and $G : \underline{K}' \rightarrow \underline{K}''$ between Hoehnke categories \underline{K} , \underline{K}' , \underline{K}'' ,

respectively, yields a *dht*-symmetric functor $FG : \underline{K} \rightarrow \underline{K}''$. The same is true for d -monoidal functors between Hoehnke categories. More precisely:

Proposition 3.7. *Let $F : \underline{K} \rightarrow \underline{K}'$ and $G : \underline{K}' \rightarrow \underline{K}''$ be d -monoidal functors between Hoehnke categories \underline{K} , \underline{K}' , \underline{K}'' . Then the functor $FG : \underline{K} \rightarrow \underline{K}''$ is a d -monoidal functor too.*

Proof. Ad (F*): Since every functor maps isomorphisms to isomorphism and

$$\begin{aligned}
(FG)^*\langle X, Y \rangle &= d_{(X \otimes Y)(FG)}^{(K'')} \left(p_1^{(K)^{X,Y}} (FG) \otimes p_2^{(K)^{X,Y}} (FG) \right) \\
&= d_{((X \otimes Y)F)G}^{(K'')} \left(\left(p_1^{(K)^{X,Y}} F \right) G \otimes \left(p_2^{(K)^{X,Y}} F \right) G \right) \\
&= \left(d_{(X \otimes Y)F}^{(K')} \right) GG^*\langle (X \otimes Y)F, (X \otimes Y)F \rangle \left(\left(p_1^{(K)^{X,Y}} F \right) G \otimes \left(p_2^{(K)^{X,Y}} F \right) G \right) \\
&= \left(d_{(X \otimes Y)F}^{(K')} \right) G \left(p_1^{(K)^{X,Y}} F \otimes p_2^{(K)^{X,Y}} F \right) GG^*\langle XF, YF \rangle \\
&= \left(d_{(X \otimes Y)F}^{(K')} \left(p_1^{(K)^{X,Y}} F \otimes p_2^{(K)^{X,Y}} F \right) \right) GG^*\langle XF, YF \rangle \\
&= (F^*\langle X, Y \rangle) GG^*\langle XF, YF \rangle
\end{aligned}$$

one obtains $(FG)^*\langle X, Y \rangle \in Iso_{K''}$.

$$\text{Ad (FI*): } t_{I(FG)}^{(K'')} = t_{(IF)G}^{(K'')} = \left(t_{IF}^{(K')} \right) Gt_{I(K')G}^{(K'')} \in Iso_{K''}$$

$$\text{since } t_{I(K')G}^{(K'')} \in Iso_{K''} \text{ and } t_{IF}^{(K')} \in Iso_{K'}.$$

$$\begin{aligned}
\text{Ad (FM*): } (\varphi \otimes \psi)(FG)(FG)^*\langle U, V \rangle &= ((\varphi \otimes \psi)F)G(F^*\langle U, V \rangle)GG^*\langle UF, VF \rangle \\
&= ((\varphi \otimes \psi)FF^*\langle U, V \rangle)GG^*\langle UF, VF \rangle \\
&= (F^*\langle X, Y \rangle)(\varphi F \otimes \psi F)GG^*\langle UF, VF \rangle \\
&= (F^*\langle X, Y \rangle)G(\varphi F \otimes \psi F)GG^*\langle UF, VF \rangle \\
&= (F^*\langle X, Y \rangle)GG^*\langle XF, YF \rangle((\varphi F)G \otimes (\psi F)G) \\
&= (FG)^*\langle X, Y \rangle(\varphi(FG) \otimes \psi(FG)). \quad \blacksquare
\end{aligned}$$

Lemma 3.8. *Let $F : \underline{K} \rightarrow \underline{K}'$ be a functor from a Hoehnke category \underline{K} into a Hoehnke category \underline{K}' such that the conditions*

$$(sFD) \quad \forall X \in |K| \left(d_X^{(K)} F = d_{XF}^{(K')} \right),$$

$$(sFT) \quad \forall X \in |K| \left(t_X^{(K)} F = t_{XF}^{(K')} \right), \text{ and}$$

$$(sFM) \quad \forall \varphi, \psi \in K \left((\varphi \otimes \psi) F = (\varphi F \otimes \psi F) \right)$$

are fulfilled.

Then F has the properties

$$(sF*) \quad \forall X, Y \in |K| (F^* \langle X, Y \rangle \in Un_{K'}) \text{ and}$$

$$(sFI^*) \quad t_{IF}^{(K')} \in Un_{K'},$$

i.e. $F : \underline{K} \rightarrow \underline{K}'$ is a strictly d -monoidal functor.

Proof. Assuming (sFT) one has $1_{IF}^{(K')} = 1_I^{(K)} F = t_I^{(K)} F = t_{IF}^{(K')}$, hence $IF = I^{(K')}$ and (sFI*) is fulfilled.

Moreover,

$$\begin{aligned} & \forall X, Y \in |K| \left(K'[(X \otimes Y)F, (X \otimes Y)F] \ni 1_{X \otimes Y}^{(K)} F = \left(1_X^{(K)} \otimes 1_Y^{(K)} \right) F \right. \\ & \left. = 1_X^{(K)} F \otimes 1_Y^{(K)} F = 1_{XF}^{(K')} \otimes 1_{YF}^{(K')} = 1_{XF \otimes YF}^{(K')} \in K'[XF \otimes YF] \right), \end{aligned}$$

hence

$$\forall X, Y \in |K| \left((X \otimes Y)F = XF \otimes YF \right).$$

Now let X and Y be any objects of $|K|$. Then

$$\begin{aligned} F^* \langle X, Y \rangle &= d_{(X \otimes Y)F}^{(K')} \left(p_1^{(K)X,Y} F \otimes p_2^{(K)X,Y} F \right) && \text{(by definition)} \\ &= d_{X \otimes Y}^{(K)} F \left(p_1^{(K)X,Y} F \otimes p_2^{(K)X,Y} F \right) && \text{((sFD))} \\ &= \left(d_{X \otimes Y}^{(K)} \left(p_1^{(K)X,Y} \otimes p_2^{(K)X,Y} \right) \right) F && \text{((sFM))} \\ &= \left(1_{X \otimes Y}^{(K)} \right) F = 1_{XF \otimes YF}^{(K')} \in Un_{K'}. && \blacksquare \end{aligned}$$

Proposition 3.9. *If functors $F : \underline{K} \rightarrow \underline{K}'$ and $G : \underline{K}' \rightarrow \underline{K}''$ between Hoehnke categories \underline{K} , \underline{K}' , \underline{K}'' fulfil the conditions (sFD), (sFT), and (sFM), then the functor $FG : \underline{K} \rightarrow \underline{K}''$ satisfies the same conditions. ■*

Corollary 3.10. *If any functor $F : \underline{K} \rightarrow \underline{K}'$ as above fulfils (sFT) and (sFM), then F is a d -monoidal functor satisfying (sFI*).*

Proof. It remains to prove the validity of (F*).

$$\begin{aligned}
F^*\langle X, Y \rangle &= d_{(X \otimes Y)F}^{(K')} \left(p_1^{(K)X,Y} F \otimes p_2^{(K)X,Y} F \right) \\
&= d_{XF \otimes YF}^{(K')} \left(\left(\left(1_X^{(K)} \otimes t_Y^{(K)} \right) r_X^{(K)} \right) F \otimes \left(\left(t_X^{(K)} \otimes 1_Y^{(K)} \right) l_X^{(K)} \right) F \right) \\
&= d_{XF \otimes YF}^{(K')} \left(\left(1_X^{(K)} F \otimes t_Y^{(K)} F \right) \otimes \left(t_X^{(K)} F \otimes 1_Y^{(K)} F \right) \right) \left(r_X^{(K)} F \otimes l_X^{(K)} F \right) \\
&= \left(d_{XF}^{(K')} \left(\left(1_X^{(K)} \otimes t_X^{(K)} \right) F \otimes d_{YF}^{(K')} \left(t_Y^{(K)} \otimes 1_Y^{(K)} \right) F \right) b_{XF,IF,IF,YF}^{(K')} \right. \\
&\qquad \qquad \qquad \left. \left(r_X^{(K)} F \otimes l_X^{(K)} F \right) \right) \\
&= \left(\left(r_{XF}^{(K')} \right)^{-1} \otimes \left(l_{YF}^{(K')} \right)^{-1} \right) 1_{(XF \otimes IF) \otimes (IF \otimes YF)}^{(K')} \left(r_X^{(K)} F \otimes l_X^{(K)} F \right) \\
&= \left(r_{XF}^{(K')} \right)^{-1} r_X^{(K)} F \otimes \left(l_{YF}^{(K')} \right)^{-1} l_X^{(K)} F \in Iso_{K'}. \quad \blacksquare
\end{aligned}$$

4. FUNCTORS BETWEEN THEORIES, THEORY MORPHISMS

The following considerations are confined to $dhts$ -theories, but it is easily to see that all results are transferable to $dhth\nabla s$ -theories and dts -theories, respectively.

Lemma 4.1. *Let F be a d -monoidal functor from a Hoehnke theory \mathbf{T} into a Hoehnke theory \mathbf{T}' such that all morphisms $\tilde{F}\langle A, B \rangle$ and i_F are central morphisms only. Then F maps every central morphism $c \in \mathbf{C}_{\mathbf{T}}$ to a central morphism $cF \in \mathbf{C}_{\mathbf{T}'}$.*

Proof. Every functor maps unit morphisms to unit morphism. Any d -monoidal functor fulfils the conditions (FA), (FR), and (FL) and since i_F

and every $\tilde{F}\langle A, B \rangle$ are central morphisms, all images $a_{A,B,C}F$, r_AF , l_AF , $(a_{A,B,C}^{-1})F$, $(r_A^{-1})F$, $(l_A^{-1})F$ are central morphisms in \mathbf{T}' .

Therefore, the images of all morphisms of $\mathbf{C}_{\mathbf{T}}^{(0)}$ are central morphisms in \mathbf{T}' .

Assuming that all morphisms of $\mathbf{C}_{\mathbf{T}}^{(n)}$ for any $n \in \mathbb{N}$ are mapped by F to central morphisms in \mathbf{T}' one has

$$\begin{aligned} \forall \varphi \in \mathbf{C}_{\mathbf{T}}^{(n+1)} \setminus \mathbf{C}_{\mathbf{T}}^{(n)} \exists \varphi_1, \varphi_2 \in \mathbf{C}_{\mathbf{T}}^{(n)} (\varphi F = (\varphi_1 \varphi_2)F = \\ (\varphi_1 F)(\varphi_2 F) \in \mathbf{C}_{\mathbf{T}'} \vee \varphi F = (\varphi_1 \otimes \varphi_2)F = \\ (\tilde{F}\langle \text{dom } \varphi_1, \text{dom } \varphi_2 \rangle)^{-1}(\varphi_1 F \otimes \varphi_2 F) \tilde{F}\langle \text{cod } \varphi_1, \text{cod } \varphi_2 \rangle) \in \mathbf{C}_{\mathbf{T}'}), \end{aligned}$$

hence $\forall \varphi \in \mathbf{C}_{\mathbf{T}} (\varphi F \in \mathbf{C}_{\mathbf{T}'})$. ■

Observe that especially strict d -monoidal functors map central morphisms to central morphisms.

Theorem 4.2. *Let \mathbf{T} be a J -sorted Hoehnke theory. Then the function Φ as defined in 2.5 induces a d -monoidal functor $\Phi : \mathbf{T} \rightarrow \mathbf{T}$ relative to $\tilde{\Phi}$ and i_{Φ} such that*

$$\forall X, Y \in |\mathbf{T}| (\tilde{\Phi}\langle X, Y \rangle := (c_X^{-1} \otimes c_Y^{-1})c_{X \otimes Y}) \quad \text{and} \quad i_{\Phi} := 1_I.$$

Proof. The object mapping is given by the function $\Phi : |\mathbf{T}| \rightarrow |\mathbf{T}|$, namely

$$X\Phi = \begin{cases} X, & \text{for all } X \in J \cup \{I, O\}, \\ I, & \text{for all } X \in \langle I \rangle, \\ \bigotimes_{j=1}^n A_j & \text{for all } X \in |\mathbf{T}| \setminus \langle I \rangle^{\circ}, \end{cases}$$

where $A_1, \dots, A_n \in J$ are exactly the factors appearing in X in this sequence independently of brackets.

Using the uniquely determined central morphisms $c_X \in \mathbf{C}_{\mathbf{T}}[X, X\Phi]$ define a morphism mapping by

$$\mathbf{T}[X, Y] \ni \varphi \mapsto \varphi\Phi := c_X^{-1} \varphi c_Y \in \mathbf{T}[X\Phi, Y\Phi].$$

Then the functor conditions are fulfilled, since

$\forall \varphi \in \mathbf{T}$ ($(\text{dom} \varphi)\Phi = \text{dom}(\varphi\Phi)$, $(\text{cod} \varphi)\Phi = \text{cod}(\varphi\Phi)$) by definition,

$$\forall X \in |\mathbf{T}| \quad (1_X\Phi = c_X^{-1}1_Xc_X = 1_{X\Phi},$$

$$\forall X, Y, P \in |\mathbf{T}| \quad \forall \varphi \in \mathbf{T}[X, Y] \quad \forall \psi \in \mathbf{T}[Y, P]$$

$$((\varphi\psi)\Phi = c_X^{-1}\varphi\psi c_P = c_X^{-1}\varphi c_Y c_Y^{-1}\psi c_P = (\varphi\Phi)(\psi\Phi)).$$

By Theorem 3.6, it is sufficient to prove the conditions (F*), (FI*), and (FM*) for the functor Φ .

Ad (F*): Let X and Y be arbitrary objects of \mathbf{T} . Then, by definition,

$$\begin{aligned} \Phi^*\langle X, Y \rangle &= d_{(X \otimes Y)\Phi}(p_1^{X,Y}\Phi \otimes p_1^{X,Y}\Phi) = d_{(X \otimes Y)\Phi}(c_{X,Y}^{-1}p_1^{X,Y}c_X \otimes c_{X,Y}^{-1}p_2^{X,Y}c_Y) \\ &= c_{X,Y}^{-1}d_{(X \otimes Y)}(p_1^{X,Y} \otimes p_2^{X,Y})(c_X \otimes c_Y) = c_{X,Y}^{-1}(c_X \otimes c_Y) \in \mathbf{C}_{\mathbf{T}} \subseteq \text{Iso}_{\mathbf{T}}. \end{aligned}$$

Ad (FI*): Because of $I\Phi = I$, $t_{I\Phi} = t_I = 1_I \in \text{Iso}_{\mathbf{T}}$.

Ad (FM*): For all objects X_1, X_2, Y_1, Y_2 and all morphisms $\varphi_i \in \mathbf{T}[X_i, Y_i]$, $i \in \{1, 2\}$, the equation

$$\begin{aligned} (\varphi_1 \otimes \varphi_2)\Phi^*\langle Y_1, Y_2 \rangle &= c_{X_1 \otimes X_2}^{-1}(\varphi_1 \otimes \varphi_2)c_{Y_1 \otimes Y_2}c_{Y_1 \otimes Y_2}^{-1}(c_{Y_1} \otimes c_{Y_2}) \\ &= c_{X_1 \otimes X_2}^{-1}(\varphi_1c_{Y_1} \otimes \varphi_2c_{Y_2}) \\ &= c_{X_1 \otimes X_2}^{-1}(c_{X_1} \otimes c_{X_2}) \left(c_{X_1}^{-1}\varphi_1c_{Y_1} \otimes c_{X_2}^{-1}\varphi_2c_{Y_2} \right) \\ &= \Phi^*\langle X_1, X_2 \rangle(\varphi_1\Phi \otimes \varphi_2\Phi) \end{aligned}$$

is valid. Therefore, $(\Phi, \tilde{\Phi}, i_\Phi)$ with $\tilde{\Phi} := (\Phi^*)^{-1}$ and $i_\Phi := 1_I$ is a d -monoidal functor from $\underline{\mathbf{T}}$ into $\underline{\mathbf{T}}$. ■

The functor Φ shall be called the canonical functor of $\underline{\mathbf{T}}$.

Corollary 4.3. *Let $\underline{\mathbf{T}}$ be a J -sorted dhts-theory. Then the canonical functor of $\underline{\mathbf{T}}$ possesses the following properties:*

- (1) $\forall X \in |\mathbf{T}| ((X\Phi)\Phi = X\Phi)$,
- (2) $\forall X \in |\mathbf{T}| ((t_X)\Phi = t_{X\Phi})$,
- (3) $\forall X \in |\mathbf{T}| ((r_X)\Phi = 1_{X\Phi} = (l_X)\Phi)$,
- (4) $\forall X \in |\mathbf{T}| (d_X\Phi\Phi^*\langle X, X \rangle = d_{X\Phi})$,
- (5) $\forall X \in |\mathbf{T}| (\nabla_X\Phi = \Phi^*\langle X, X \rangle\nabla_{X\Phi})$,
- (6) $\forall X \in |\mathbf{T}| (\Phi^*\langle X, I \rangle = (r_{X\Phi})^{-1}, \Phi^*\langle I, X \rangle = (l_{X\Phi})^{-1})$,
- (7) $\forall X \in |\mathbf{T}| ((c_X)\Phi = 1_{X\Phi} = (1_X)\Phi = c_{X\Phi})$,
- (8) $\forall \varphi \in \mathbf{T} \left(\text{dom}\varphi = \bigotimes_{j=1}^n A_j \wedge \text{cod}\varphi = \bigotimes_{k=1}^m B_k \wedge A_j, B_k \in J \Rightarrow \varphi\Phi = \varphi \right)$,
- (9) $\forall \varphi \in \mathbf{T} ((\varphi\Phi)\Phi = \varphi\Phi)$.

Proof. Ad (1): $(X\Phi)\Phi = \left(\bigotimes_{j=1}^n A_j \right) \Phi = \bigotimes_{j=1}^n A_j = X\Phi$.

Ad (2): $(t_X)\Phi = c_X^{-1}t_Xc_I = t_{X\Phi}$ since $c_X \in \text{Iso}_T \wedge c_I = 1_I$.

Ad (3): The assertion is a special case of (7).

Ad (4): $d_X\Phi = c_X^{-1}d_Xc_{X\otimes X} = d_{X\Phi} (c_X^{-1} \otimes c_X^{-1}) c_{X\otimes X} = d_{X\Phi}(\Phi^*\langle X, X \rangle)^{-1}$
 $\Rightarrow d_X\Phi\Phi^*\langle X, X \rangle = d_{X\Phi}$.

Ad (5): $\nabla_X\Phi = (c_{X\otimes X})^{-1}\nabla_Xc_X = (c_{X\otimes X})^{-1}(c_X \otimes c_X)\nabla_{X\Phi} = \Phi^*\langle X, X \rangle\nabla_{X\Phi}$.

Ad (6): $\Phi^*\langle X, I \rangle \in \mathbf{C}_T[X\Phi, X\Phi \otimes I]$ and $r_{X\Phi} \in \mathbf{C}_T[X\Phi \otimes I, X\Phi]$,

hence $\Phi^*\langle X, I \rangle = (r_{X\Phi})^{-1}$ by the coherence principle.

Ad (7): $c_X \in \mathbf{C}_T[X, X\Phi] \Rightarrow (c_X)\Phi \in \mathbf{C}_T[X\Phi, (X\Phi)\Phi] = \mathbf{C}_T[X\Phi, X\Phi] \ni 1_{X\Phi}$

$$\Rightarrow (c_X)\Phi = 1_{X\Phi} = 1_{X\Phi}.$$

$c_{X\Phi} \in \mathbf{C}_T[X\Phi, X\Phi\Phi] = \mathbf{C}_T[X\Phi, X\Phi]$ shows $c_{X\Phi} = 1_{X\Phi}$.

Ad (8): $\varphi\Phi = c_{X\Phi}^{-1}\varphi c_{Y\Phi} = \varphi$, where $X = \bigotimes_{j=1}^n A_j = X\Phi \wedge Y =$

$$= \bigotimes_{k=1}^m B_k = Y\Phi.$$

Ad (9): $(\varphi\Phi)\Phi = (c_X^{-1}\varphi c_Y)\Phi = (c_X^{-1})\Phi(\varphi)\Phi(c_Y)\Phi = \varphi\Phi$. ■

Definition 4.4. Let \mathbf{T} be a J -sorted Hoehnke theory and let Φ be the canonical d -monoidal functor of \mathbf{T} . Then define a binary relation \varkappa for objects and morphisms of \mathbf{T} as follows:

$$(X, Y) \in \varkappa :\Leftrightarrow X\Phi = Y\Phi,$$

$$(\varphi_1, \varphi_2) \in \varkappa :\Leftrightarrow \varphi_1\Phi = \varphi_2\Phi.$$

Theorem 4.5. *The relation \varkappa defined by the canonical d -monoidal functor Φ of a J -sorted Hoehnke theory \mathbf{T} as above is a “generalized” congruence on \mathbf{T} .*

Proof. Considering small categories as many-sorted total algebras, a congruence ρ is defined as a family of equivalence relations on the isolated morphism sets, i.e. $(\varphi, \psi) \in \rho \Rightarrow \text{dom}\varphi = \text{dom}\psi \wedge \text{cod}\varphi = \text{cod}\psi$.

That is not true for the relation \varkappa , since only $\forall \varphi, \psi \in \mathbf{T}((\varphi, \psi) \in \varkappa \Rightarrow (\text{dom}\varphi)\Phi = (\text{dom}\psi)\Phi \wedge (\text{cod}\varphi)\Phi = (\text{cod}\psi)\Phi)$, because of

$$(\varphi, \psi) \in \varkappa \Rightarrow \varphi\Phi = \psi\Phi \Rightarrow c_{\text{dom}\varphi}^{-1}\varphi c_{\text{cod}\varphi} = c_{\text{dom}\psi}^{-1}\psi c_{\text{cod}\psi}$$

$$\Rightarrow (\text{dom}\varphi)\Phi = (\text{dom}\psi)\Phi \wedge (\text{cod}\varphi)\Phi = (\text{cod}\psi)\Phi.$$

Moreover, the relation \varkappa is not compatible with the morphism composition in the strong sense.

By definition, the relation \varkappa is reflexive, symmetric, and transitive for objects and morphisms, respectively.

The relation is compatible with \otimes -operation of morphisms and objects, respectively, because of the following argumentation.

Using of (FM*) and Corollary 4.3 (5) one has for morphisms:

$$\begin{aligned}
(\varphi_1, \varphi_2), (\psi_1, \psi_2) \in \varkappa &\Rightarrow (\varphi_1 \otimes \psi_1)\Phi = \Phi^*\langle X_1, P_1 \rangle (\varphi_1 \Phi \otimes \psi_1 \Phi) (\Phi^*\langle Y_1, Q_1 \rangle)^{-1} \\
&= \Phi^*\langle X_1, P_1 \rangle (\varphi_2 \Phi \otimes \psi_2 \Phi) (\Phi^*\langle Y_1, Q_1 \rangle)^{-1} \\
&= c_{X_1 \otimes P_1}^{-1} \left(c_{X_1} c_{X_2}^{-1} \otimes c_{P_1} c_{P_2}^{-1} \right) (\varphi_2 \otimes \psi_2) \left(c_{Y_2} c_{Y_1}^{-1} \otimes c_{Q_2} c_{Q_1}^{-1} \right) c_{Y_1 \otimes Q_1} \\
\Rightarrow (\varphi_1 \otimes \psi_1)\Phi &= ((\varphi_1 \otimes \psi_1)\Phi)\Phi \\
&= \left(c_{X_1 \otimes P_1}^{-1} \left(c_{X_1} c_{X_2}^{-1} \otimes c_{P_1} c_{P_2}^{-1} \right) (\varphi_2 \otimes \psi_2) \left(c_{Y_2} c_{Y_1}^{-1} \otimes c_{Q_2} c_{Q_1}^{-1} \right) c_{Y_1 \otimes Q_1} \right) \Phi \\
&= (\varphi_2 \otimes \psi_2)\Phi \\
\Rightarrow (\varphi_1 \otimes \psi_1, \varphi_2 \otimes \psi_2) &\in \varkappa.
\end{aligned}$$

Concerning the object relation one obtains

$$\begin{aligned}
(X_1, X_2) \in \varkappa \wedge (Y_1, Y_2) \in \varkappa &\Rightarrow X_1\Phi = X_2\Phi \wedge Y_1\Phi = Y_2\Phi \\
\Rightarrow 1_{X_1}\Phi &= 1_{X_2}\Phi \wedge 1_{Y_1}\Phi = 1_{Y_2}\Phi \\
\Rightarrow (1_{X_1}, 1_{X_2}) \in \varkappa &\wedge (1_{Y_1}, 1_{Y_2}) \in \varkappa \\
\Rightarrow (1_{X_1} \otimes 1_{Y_1}, 1_{X_2} \otimes 1_{Y_2}) \in \varkappa \\
\Rightarrow (1_{X_1 \otimes Y_1}, 1_{X_2 \otimes Y_2}) \in \varkappa \\
\Rightarrow 1_{X_1 \otimes Y_1}\Phi &= 1_{X_2 \otimes Y_2}\Phi \\
\Rightarrow 1_{(X_1 \otimes Y_1)\Phi} &= 1_{(X_2 \otimes Y_2)\Phi} \\
\Rightarrow (X_1 \otimes Y_1)\Phi &= (X_2 \otimes Y_2)\Phi \\
\Rightarrow (X_1 \otimes Y_1, X_2 \otimes Y_2) \in \varkappa.
\end{aligned}$$

The relation \varkappa is, as already mentioned, reflexive, therefore it preserves all morphisms which are determined by constant operation symbols.

For the morphism composition:

Let $\varphi_i \in \mathbf{T}[X_i, Y_i], \psi_i \in \mathbf{T}[P_i, Q_i]$ for $i \in \{1, 2\}$ be arbitrary morphisms of $\underline{\mathbf{T}}$. Then, for $Y_1 = P_1$, i. e. φ_1 is composable with ψ_1 ,

$$\begin{aligned} (\varphi_1, \varphi_2), (\psi_1, \psi_2) \in \varkappa &\Rightarrow (\varphi_1 \psi_1) \Phi = (\varphi_1 \Phi)(\psi_1 \Phi) \\ &= (\varphi_2 \Phi)(\psi_2 \Phi) = c_{X_2}^{-1} \varphi_2 c_{Y_2} c_{P_2}^{-1} \psi_2 c_{Q_2}, \end{aligned}$$

therefore, by Corollary 4.3 (7) and (5),

$$(\varphi_1 \psi_1) \Phi = ((\varphi_1 \psi_1) \Phi) \Phi = \left(c_{X_2}^{-1} \varphi_2 c_{Y_2} c_{P_2}^{-1} \psi_2 c_{Q_2} \right) \Phi = (\varphi_2 c_{Y_2, P_2} \psi_2) \Phi,$$

hence $(\varphi_1 \psi_1, \varphi_2 c_{Y_2, P_2} \psi_2) \in \varkappa$.

Observe that especially φ_2 and ψ_2 have not to be composable in general, but there is a central morphism c such that there exists the compositum $\varphi_2 c \psi_2$. \blacksquare

Remark. It is easy to verify that the generating central morphisms $1, a, a^{-1}, r, r^{-1}, l, l^{-1}$ of any J -sorted theory $\underline{\mathbf{T}}$ fulfil even the following conditions:

$$\begin{aligned} \forall X, Y, P \in |\mathbf{T}| \quad &((1_{X \otimes (Y \otimes P)}, 1_{(X \otimes Y) \otimes P}) \in \varkappa), \\ \forall X, Y, P \in |\mathbf{T}| \quad &((a_{X, Y, P}, 1_{X \otimes (Y \otimes P)}) \in \varkappa \wedge ((a_{X, Y, P})^{-1}, 1_{(X \otimes Y) \otimes P}) \in \varkappa), \\ \forall X \in |\mathbf{T}| \quad &((1_{X \otimes I}, 1_X), (1_{I \otimes X}, 1_X) \in \varkappa), \\ \forall X \in |\mathbf{T}| \quad &((r_X, 1_{X \otimes I}), ((r_X)^{-1}, 1_X), ((l_X), 1_{I \otimes X}), ((l_X)^{-1}, 1_X) \in \varkappa). \end{aligned}$$

Theorem 4.6. *To every J -sorted Hoehnke theory*

$$\underline{\mathbf{T}} \in |\mathit{Th}_{dht}^{\circ}(J)|$$

there exists in a natural manner a J -sorted strict Hoehnke theory

$$\mathbf{T}_s \in |sTh_{dht}^\circ(J)|.$$

Proof. The canonical d -monoidal functor $\Phi : \mathbf{T} \rightarrow \mathbf{T}$ related to any J -sorted Hoehnke theory \mathbf{T} induces the “generalized” congruence \varkappa .

Construct a new category \mathbf{T}_s by using the knowledge about $\underline{H}^\circ, \underline{S}^\circ$ and the functions W and W^* .

$$|\mathbf{T}_s| := S^\circ \quad (:= S),$$

$$\mathbf{T}_s := \{[\varphi]_\varkappa \mid \varphi \in \mathbf{T}\}, \text{ where } [\varphi]_\varkappa = \{\varphi' \in \mathbf{T} \mid \varphi\Phi = \varphi'\Phi\},$$

$$\text{dom}^{(\mathbf{T}_s)}[\varphi]_\varkappa := \left(\text{dom}^{(\mathbf{T})}\varphi\right)W^*, \text{ cod}^{(\mathbf{T}_s)}[\varphi]_\varkappa := \left(\text{cod}^{(\mathbf{T})}\varphi\right)W^*,$$

$$1_A^{(\mathbf{T}_s)} := \left[1_{AW}^{(\mathbf{T})}\right]_\varkappa,$$

$$[\varphi]_\varkappa \cdot_{(\mathbf{T}_s)} [\psi]_\varkappa := [\varphi_{CY,P}\psi]_\varkappa, \text{ where } Y\Phi = (\text{cod}\varphi)\Phi = (\text{dom}\psi)\Phi = P\Phi$$

$$(\Leftrightarrow YW^* = (\text{cod}\varphi)W^* = (\text{dom}\psi)W^*) = PW^*),$$

$$A \otimes_{(\mathbf{T}_s)} B = (AW \otimes_{(\mathbf{T})} BW)W^* \text{ (by (W4))},$$

$$[\varphi]_\varkappa \otimes_{(\mathbf{T}_s)} [\psi]_\varkappa := [\varphi \otimes_{(\mathbf{T})} \psi]_\varkappa,$$

$$a_{A,B,C}^{(\mathbf{T}_s)} := \left[a_{AW,BW,CW}^{(\mathbf{T})}\right]_\varkappa = \left[1_{AW \otimes (BW \otimes CW)}^{(\mathbf{T})}\right]_\varkappa,$$

$$r_A^{(\mathbf{T}_s)} := \left[r_{AW}^{(\mathbf{T})}\right]_\varkappa = \left[1_{AW}^{(\mathbf{T})}\right]_\varkappa = \left[l_{AW}^{(\mathbf{T})}\right]_\varkappa =: l_A^{(\mathbf{T}_s)},$$

$$s_{A,B}^{(\mathbf{T}_s)} := \left[s_{AW,BW}^{(\mathbf{T})}\right]_\varkappa, \quad d_A^{(\mathbf{T}_s)} := \left[d_{AW}^{(\mathbf{T})}\right]_\varkappa, \quad t_A^{(\mathbf{T}_s)} := \left[t_{AW}^{(\mathbf{T})}\right]_\varkappa, \quad \nabla_A^{(\mathbf{T}_s)} := \left[\nabla_{AW}^{(\mathbf{T})}\right]_\varkappa,$$

$$o^{(\mathbf{T}_s)} := \left[o^{(\mathbf{T})}\right]_\varkappa.$$

Obviously, $(S^\circ; \otimes, I, O)$ is an algebra of type $(2, 0, 0)$ with an associative binary operation, a unit element I , and a zero element O .

Moreover, $(|\mathbf{T}_s|, \mathbf{T}_s, \cdot, \text{dom}, \text{cod}, 1)$ is a small category, since $|\mathbf{T}_s|$ is a set and

$$\begin{aligned}
[\varphi]_{\varkappa} \in \mathbf{T}_s[A, B] &\Rightarrow \varphi \in \mathbf{T}[X, Y] \wedge A = XW^*, B = YW^* \Rightarrow 1_A[\varphi]_{\varkappa} \\
&= [1_X]_{\varkappa}[\varphi]_{\varkappa} = [1_X c_{X, X} \varphi]_{\varkappa} = [\varphi]_{\varkappa} = [\varphi c_{Y, Y} 1_Y]_{\varkappa} = [\varphi]_{\varkappa} [1_Y]_{\varkappa} = [\varphi]_{\varkappa} 1_B, \\
[\varphi]_{\varkappa} \in \mathbf{T}_s[A, B], [\psi]_{\varkappa} \in \mathbf{T}_s[B, C], [\chi]_{\varkappa} \in \mathbf{T}_s[C, D] \\
&\Rightarrow [\varphi]_{\varkappa}([\psi]_{\varkappa}[\chi]_{\varkappa}) = [\varphi]_{\varkappa}[\psi c_{P, Q} \chi]_{\varkappa} = [\varphi c_{X, Y} \psi c_{P, Q} \chi]_{\varkappa} \\
&= [\varphi c_{X, Y} \psi]_{\varkappa}[\chi]_{\varkappa} = ([\varphi]_{\varkappa}[\psi]_{\varkappa})[\chi]_{\varkappa}.
\end{aligned}$$

$(\mathbf{T}_s; \otimes, I, 1, 1, s)$ is a symmetric strictly monoidal category since the defining conditions are fulfilled. Observe that to every morphism $\rho \in \mathbf{T}_s[A, B]$ there is a morphism $\varphi \in \mathbf{T}[X, Y]$ such that $A = XW^*, B = YW^*, \rho = [\varphi]_{\varkappa}$.

Ad (F1): $\forall \rho, \rho' \in \mathbf{T}_s$ ($\text{dom}(\rho \otimes \rho') = \text{dom}([\varphi]_{\varkappa} \otimes [\varphi']_{\varkappa})$)

$$\begin{aligned}
&= \text{dom}[\varphi \otimes \varphi']_{\varkappa} = (\text{dom}(\varphi \otimes \varphi'))W^* \\
&= ((\text{dom} \varphi) \otimes (\text{dom} \varphi'))W^* = (\text{dom} \varphi)W^* \otimes (\text{dom} \varphi')W^* \\
&= (\text{dom} [\varphi]_{\varkappa}) \otimes (\text{dom} [\varphi']_{\varkappa}) = \text{dom} \rho \otimes \text{dom} \rho'.
\end{aligned}$$

Ad (F2): The assertion $\forall \rho, \rho' \in \mathbf{T}_s$ ($\text{cod}(\rho \otimes \rho') = \text{cod} \rho \otimes \text{cod} \rho'$) will be proved in the same manner.

Ad (F3): $\forall A, B \in |\mathbf{T}_s|$ ($1_{A \otimes B} = [1_{(A \otimes B)W}]_{\varkappa} = [1_{AW \otimes BW}]_{\varkappa}$)

$$= [1_{AW} \otimes 1_{BW}]_{\varkappa} = [1_{AW}]_{\varkappa} \otimes [1_{BW}]_{\varkappa} = 1_A \otimes 1_B,$$

since $\underline{\mathbf{T}}$ is a symmetric monoidal category and for all $A, B \in S^\circ$ one has $(A \otimes B)W\Phi = (A \otimes B)WW^*W = (A \otimes B)W = (AWW^* \otimes BWW^*)W = (AW \otimes BW)\Phi$.

Ad (F4): $\forall A, B, C, A', B', C' \in |\mathbf{T}_s| \forall \rho \in \mathbf{T}_s[A, B]$

$$\forall \sigma \in \mathbf{T}_s[B, C] \forall \rho' \in \mathbf{T}_s[A', B'] \forall \sigma' \in \mathbf{T}_s[B', C']$$

$$\begin{aligned}
((\rho \otimes \rho')(\sigma \otimes \sigma')) &= ([\varphi]_{\varkappa} \otimes [\varphi']_{\varkappa})([\psi]_{\varkappa} \otimes [\psi']_{\varkappa}) \\
&= [\varphi \otimes \varphi']_{\varkappa}[\psi \otimes \psi']_{\varkappa} \\
&= [(\varphi \otimes \varphi')c_{Y \otimes Y', P \otimes P'}(\psi \otimes \psi')]_{\varkappa} \\
&= [(\varphi \otimes \varphi')(c_{Y, P} \otimes c_{Y', P'}) (\psi \otimes \psi')]_{\varkappa} \\
&= [\varphi c_{Y, P} \psi \otimes \varphi' c_{Y', P'} \psi']_{\varkappa} \\
&= [\varphi c_{Y, P} \psi]_{\varkappa} \otimes [\varphi' c_{Y', P'} \psi']_{\varkappa} \\
&= [\varphi]_{\varkappa}[\psi]_{\varkappa} \otimes [\varphi']_{\varkappa}[\psi']_{\varkappa} \\
&= \rho\sigma \otimes \rho'\sigma'.
\end{aligned}$$

Ad (M1), (M2), (M3): The conditions are trivially fulfilled since a and r consist of unit morphisms only.

$$\begin{aligned}
\text{Ad (M4): } \forall A, B \in |\mathbf{T}_s| \quad & \left(s_{A, B}^{(\mathbf{T}_s)} s_{B, A}^{(\mathbf{T}_s)} = \left[s_{AW, BW}^{(\mathbf{T})} \right]_{\varkappa} \left[s_{BW, AW}^{(\mathbf{T})} \right]_{\varkappa} \right. \\
&= \left[s_{AW, BW}^{(\mathbf{T})} c_{BW \otimes AW, BW \otimes AW} s_{BW, AW}^{(\mathbf{T})} \right]_{\varkappa} \\
&= \left[s_{AW, BW}^{(\mathbf{T})} s_{BW, AW}^{(\mathbf{T})} \right]_{\varkappa} = \left[1_{AW \otimes BW}^{(\mathbf{T})} \right]_{\varkappa} = 1_{(AW \otimes BW)W^*} = 1_{A \otimes B}^{(\mathbf{T}_s)}.
\end{aligned}$$

$$\begin{aligned}
\text{Ad (M5): } \forall A \in |\mathbf{T}_s| \quad & \left(s_{A, I}^{(\mathbf{T}_s)} l_A^{(\mathbf{T}_s)} = \left[s_{AW, IW}^{(\mathbf{T})} \right]_{\varkappa} \left[l_{AW}^{(\mathbf{T})} \right]_{\varkappa} = \left[s_{AW, IW}^{(\mathbf{T})} l_{AW}^{(\mathbf{T})} \right]_{\varkappa} \right. \\
&= \left[r_{AW}^{(\mathbf{T})} \right]_{\varkappa} = r_A^{(\mathbf{T}_s)} = 1_A^{(\mathbf{T}_s)}.
\end{aligned}$$

Ad (M6): $\forall A, B, C, A', B', C' \in |\mathbf{T}_s| \quad \forall \rho \in \mathbf{T}_s[A, A']$

$$\forall \sigma \in \mathbf{T}_s[B, B'] \quad \forall \tau \in \mathbf{T}_s[C, C']$$

$$\left(a_{A, B, C}^{(\mathbf{T}_s)}((\rho \otimes \sigma) \otimes \tau) = \left[a_{X, Y, P}^{(\mathbf{T})} \right]_{\varkappa} (([\varphi]_{\varkappa} \otimes [\psi]_{\varkappa}) \otimes [\chi]_{\varkappa}) \right)$$

$$\begin{aligned}
&= \left[a_{X,Y,P}^{(\mathbf{T})} c_{(X \otimes Y) \otimes P, (X \otimes Y) \otimes P} ((\varphi \otimes \psi) \otimes \chi) \right]_{\varkappa} \\
&= \left[(\varphi \otimes (\psi \otimes \chi)) a_{X',Y',P'}^{(\mathbf{T})} \right]_{\varkappa} \\
&= \left[(\varphi \otimes (\psi \otimes \chi)) c_{X' \otimes (Y' \otimes P'), X' \otimes (Y' \otimes P')} a_{X',Y',P'}^{(\mathbf{T})} \right]_{\varkappa} \\
&= \left([\varphi]_{\varkappa} \otimes ([\psi]_{\varkappa} \otimes [\chi]_{\varkappa}) \right) \left[a_{X',Y',P'}^{(\mathbf{T})} \right]_{\varkappa} \\
&= (\rho \otimes (\sigma \otimes \tau)) a_{A',B',C'}^{(\mathbf{T}_s)}.
\end{aligned}$$

Ad (M7): $\forall A, A' \in |\mathbf{T}_s| \forall \rho \in \mathbf{T}_s[A, A'] \left(r_A^{(\mathbf{T}_s)} \rho = \left[r_{AW}^{(\mathbf{T})} \right]_{\varkappa} [\varphi]_{\varkappa} \right)$

$$\begin{aligned}
&= \left[r_{AW}^{(\mathbf{T})} c_{AW,X} \varphi \right]_{\varkappa} \quad (\text{by } XW^* = AWW^* = A) \\
&= \left[\left(c_{AW,X} \varphi \otimes \mathbf{1}_I^{(\mathbf{T})} \right) r_{X'}^{(\mathbf{T})} \right]_{\varkappa} \\
&= \left[\left(c_{AW,X} \varphi \otimes \mathbf{1}_I^{(\mathbf{T})} \right) c_{X' \otimes I, X' \otimes I} r_{X'}^{(\mathbf{T})} \right]_{\varkappa} \\
&= \left[c_{AW,X} \varphi \otimes \mathbf{1}_I^{(\mathbf{T})} \right]_{\varkappa} \left[r_{X'}^{(\mathbf{T})} \right]_{\varkappa} \\
&= \left([c_{AW,X} \varphi]_{\varkappa} \otimes [\mathbf{1}_I^{(\mathbf{T})}]_{\varkappa} \right) \left[r_{X'}^{(\mathbf{T})} \right]_{\varkappa} \\
&= \left([\varphi]_{\varkappa} \otimes [\mathbf{1}_I^{(\mathbf{T})}]_{\varkappa} \right) \left[r_{X'}^{(\mathbf{T})} \right]_{\varkappa} \\
&= \left(\rho \otimes \mathbf{1}_I^{(\mathbf{T}_s)} \right) r_{A'}^{(\mathbf{T}_s)} \quad (\text{by } X'W^* = A'WW^* = A').
\end{aligned}$$

Ad (M8): $\forall A, B \in |\mathbf{T}_s| \forall \rho \in \mathbf{T}_s[A, A'], \sigma \in \mathbf{T}_s[B, B']$

$$\begin{aligned}
&\left(s_{A,B}^{(\mathbf{T}_s)} (\sigma \otimes \rho) = \left[s_{AW,BW}^{(\mathbf{T})} \right]_{\varkappa} ([\psi]_{\varkappa} \otimes [\varphi]_{\varkappa}) \right) \\
&= \left[s_{AW,BW}^{(\mathbf{T})} \right]_{\varkappa} [\psi \otimes \varphi]_{\varkappa} \\
&= \left[s_{AW,BW}^{(\mathbf{T})} c_{BW \otimes AW, Y \otimes X} (\psi \otimes \varphi) \right]_{\varkappa} \\
&= \left[c_{AW \otimes BW, X \otimes Y} s_{X,Y}^{(\mathbf{T})} (\psi \otimes \varphi) \right]_{\varkappa}
\end{aligned}$$

$$\begin{aligned}
&= \left[c_{AW \otimes BW, X \otimes Y} (\varphi \otimes \psi) s_{X', Y'}^{(\mathbf{T})} \right]_{\varkappa} \\
&= \left[(\varphi \otimes \psi) c_{X' \otimes Y', X' \otimes Y'} s_{X', Y'}^{(\mathbf{T})} \right]_{\varkappa} \\
&= [\varphi \otimes \psi]_{\varkappa} \left[s_{X', Y'}^{(\mathbf{T})} \right]_{\varkappa} \\
&= ([\varphi]_{\varkappa} \otimes [\psi]_{\varkappa}) \left[s_{X', Y'}^{(\mathbf{T})} \right]_{\varkappa} \\
&= (\rho \otimes \sigma) s_{A', B'}^{(\mathbf{T}_s)},
\end{aligned}$$

where

$$\begin{aligned}
XW^* &= AWW^* = A, & X'W^* &= A'WW^* = A', \\
YW^* &= BWW^* = B, & Y'W^* &= B'WW^* = B'.
\end{aligned}$$

■

Theorem 4.7. *Let $\mathbf{T} \in |\mathit{Th}_{dht}^{\circ}(J)|$ be a J -sorted Hoehnke theory. Then there exists in a natural manner a strictly d -monoidal functor Ψ into the corresponding strict Hoehnke theory $\mathbf{T}_s \in |\mathit{sTh}_{dht}^{\circ}(J)|$.*

Proof. Defining $X\Psi := XW^*$, $\varphi\Psi := [\varphi]_{\varkappa}$ one obtains for arbitrary objects X, Y, P and morphisms $\varphi \in T[X, Y]$, $\psi \in T[Y, P]$

$$\begin{aligned}
(\mathrm{dom}^{(T)}\varphi)\Psi &= X\Psi = XW^* = \mathrm{dom}^{(T_s)}[\varphi]_{\varkappa} = \mathrm{dom}^{(T_s)}(\varphi\Psi), \\
(\mathrm{cod}^{(T)}\varphi)\Psi &= Y\Psi = YW^* = \mathrm{cod}^{(T_s)}[\varphi]_{\varkappa} = \mathrm{cod}^{(T_s)}(\varphi\Psi), \\
1_X^{(T)}\Psi &= \left[1_X^{(T)} \right]_{\varkappa} = 1_{XW^*}^{(T_s)} = 1_{X\Psi}^{(T_s)}, \\
(\varphi \cdot_{(T)} \psi)\Psi &= [\varphi \cdot_{(T)} \psi]_{\varkappa} = [\varphi]_{\varkappa} \cdot_{(T_s)} [\psi]_{\varkappa} = (\varphi\Psi) \cdot_{(T_s)} \psi\Psi,
\end{aligned}$$

hence Ψ is a functor.

By Lemma 3.8, it is sufficient to show (sFD), (sFT), and (sFM).

$$\text{Ad (sFD):} \quad d_X^{(\mathbf{T})}\Psi = \left[d_X^{(\mathbf{T})} \right]_{\varkappa} = \left[d_{XW^*W}^{(\mathbf{T})} \right]_{\varkappa} = d_{XW^*}^{(\mathbf{T}_s)} = d_{X\Psi}^{(\mathbf{T}_s)}.$$

$$\text{Ad (sFT): } t_X^{(\mathbf{T})}\Psi = \left[t_X^{(\mathbf{T})} \right]_{\mathcal{X}} = \left[t_{XW^*W}^{(\mathbf{T})} \right]_{\mathcal{X}} = t_{XW^*}^{(\mathbf{T}_s)} = t_{X\Psi}^{(\mathbf{T}_s)}.$$

$$\text{Ad (sFM): } (\varphi \otimes \psi)\Psi = [\varphi \otimes \psi]_{\mathcal{X}} = [\varphi]_{\mathcal{X}} \otimes [\psi]_{\mathcal{X}} = \varphi\Psi \otimes \psi\Psi.$$

Therefore, $\Psi : \underline{T} \rightarrow \underline{T}_s$ is a strictly d -monoidal functor. \blacksquare

The converse question is also positively answered by the following theorem:

Theorem 4.8. *Let $\underline{\mathbf{T}}_s \in |sT_{dht}^\circ(J)|$ be a strict J -sorted Hoehnke theory. Then there corresponds to $\underline{\mathbf{T}}_s$ in a natural way a J -sorted Hoehnke theory $\underline{\mathbf{T}} \in |T_{dht}^\circ(J)|$.*

Proof. Take $|\mathbf{T}| = H^\circ$ ($|T| = H$), where $(H^\circ; \otimes, I, O)$ ($(H; \otimes, I)$) is the free \mathcal{G}° -algebra (free \mathcal{G} -algebra) freely generated by J .

Defining $\mathbf{T}[X, Y] := \{(X, \varphi, Y) \mid \varphi \in \mathbf{T}_s[XW^*, YW^*]\}$ for arbitrary $X, Y \in H^\circ$ ($X, Y \in H$) one obtains obviously $\mathbf{T}[X, Y] \cup \mathbf{T}[X', Y'] = \emptyset$ if $X \neq X'$ or $Y \neq Y'$ and, by definition, $\text{dom}^{(\mathbf{T})}(X, \varphi, Y) = X$, $\text{cod}^{(\mathbf{T})}(X, \varphi, Y) = Y$ and $1_X^{(\mathbf{T})} = (X, 1_{XW^*}^{(\mathbf{T}_s)}, X)$.

Morphisms (X, φ, Y) and (P, ψ, Q) are composable for $Y = P$ defined by

$$(X, \varphi, Y) \cdot_{(\mathbf{T})} (Y, \psi, Q) := (X, \varphi \cdot_{(\mathbf{T}_s)} \psi, Q).$$

Then

$$1_X^{(\mathbf{T})} \cdot_{(\mathbf{T})} (X, \varphi, Y) = (X, 1_{XW^*}^{(\mathbf{T}_s)}, X) \cdot_{(\mathbf{T})} (X, \varphi, Y) = (X, 1_{XW^*}^{(\mathbf{T}_s)} \varphi, Y) = (X, \varphi, Y),$$

$$(X, \varphi, Y) \cdot_{(\mathbf{T})} 1_Y^{(\mathbf{T})} = (X, \varphi, Y) \cdot_{(\mathbf{T})} (Y, 1_{YW^*}^{(\mathbf{T}_s)}, Y) = (X, \varphi 1_{YW^*}^{(\mathbf{T}_s)}, Y) = (X, \varphi, Y),$$

$$(X, \varphi, Y) \cdot_{(\mathbf{T})} ((Y, \psi, P) \cdot_{(\mathbf{T})} (P, \chi, Q)) = (X, \varphi(\psi\chi), Q)$$

$$= (X, (\varphi\psi)\chi, Q) = ((X, \varphi, Y) \cdot_{(\mathbf{T})} (Y, \psi, P)) \cdot_{(\mathbf{T})} (P, \chi, Q),$$

hence one has a category.

By the agreements

$$(X_1, \varphi_1, Y_1) \otimes_{(\mathbf{T})} (X_2, \varphi_2, Y_2) := (X_1 \otimes_{(\mathbf{T}_s)} X_2, \varphi_1 \otimes_{(\mathbf{T}_s)} \varphi_2, Y_1 \otimes_{(\mathbf{T}_s)} Y_2),$$

$$a_{X,Y,P}^{(\mathbf{T})} := (X \otimes (Y \otimes P), 1_{XW^* \otimes YW^* \otimes PW^*}^{(\mathbf{T}_s)}, (X \otimes Y) \otimes P),$$

$$r_X^{(\mathbf{T})} := (X \otimes I, 1_{XW^*}^{(\mathbf{T}_s)}, X),$$

$$l_X^{(\mathbf{T})} := (I \otimes X, 1_{XW^*}^{(\mathbf{T}_s)}, X),$$

$$s_{X,Y}^{(\mathbf{T})} := (X \otimes Y, s_{XW^* \otimes YW^*}^{(\mathbf{T}_s)}, Y \otimes X),$$

$$d_X^{(\mathbf{T})} := (X, d_{XW^*}^{(\mathbf{T}_s)}, X \otimes X),$$

$$t_X^{(\mathbf{T})} := (X, t_{XW^*}^{(\mathbf{T}_s)}, I),$$

$$o^{(\mathbf{T})} := (I, o^{(\mathbf{T}_s)}, O)$$

one obtains a *dhts*-category $(\mathbf{T}, \otimes_{(\mathbf{T})}, I, a^{(\mathbf{T})}, r^{(\mathbf{T})}, l^{(\mathbf{T})}, s^{(\mathbf{T})}, t^{(\mathbf{T})}, o^{(\mathbf{T})})$, i.e. a Hoehnke theory in $|T_{dht}^\circ(J)|$, since the validity of the defining axioms obviously carries over from $\underline{\mathbf{T}}_s$ into $\underline{\mathbf{T}}$. ■

Remark. If $\underline{\mathbf{T}}_s \in |sT_{dht\triangledown}^\circ(J)|$ is even any strict J -sorted Hoehnke theory with halfdiagonalinversions, then one obtains by the additional agreement

$$\nabla_X^{(\mathbf{T})} := (X \otimes X, \nabla_{XW^*}^{(\mathbf{T}_s)}, X)$$

a *dht\triangledown s*-category $(\mathbf{T}, \otimes_{(\mathbf{T})}, I, a^{(\mathbf{T})}, r^{(\mathbf{T})}, l^{(\mathbf{T})}, s^{(\mathbf{T})}, t^{(\mathbf{T})}, \nabla^{(\mathbf{T})}, o^{(\mathbf{T})})$, i.e. a Hoehnke theory in $|T_{dht\triangledown}^\circ(J)|$.

Definition 4.9. Let $\underline{\mathbf{T}}$ and $\underline{\mathbf{T}}'$ be J -sorted Hoehnke theories in $|Th_{dht}^\circ(J)|$ and $|sTh_{dht}^\circ(J)|$, respectively.

Then a d -monoidal functor $F : \underline{\mathbf{T}} \rightarrow \underline{\mathbf{T}}'$ is called *theory morphism*, if, in addition, the conditions

$$(Th1) \quad \forall X \in |\mathbf{T}| \quad (XF = X),$$

$$(sF*) \quad \forall X, Y \in |\mathbf{T}| \quad (\tilde{F}\langle X, Y \rangle \in Un_{K'})$$

are fulfilled.

Lemma 4.10. *Every theory morphism $F : \underline{\mathbf{T}} \rightarrow \underline{\mathbf{T}}'$ has the properties (sFD), (sFT), (sFM), (sFI*).*

Conversely, any functor $F : \underline{\mathbf{T}} \rightarrow \underline{\mathbf{T}}'$ is a theory morphism between J -sorted Hoehnke theories $\underline{\mathbf{T}}$ and $\underline{\mathbf{T}}'$, whenever F satisfies (Th1), (sFD), (sFT), and (sFM).

Proof. The assertion is an immediate consequence of Lemma 3.8 and Corollary 3.10. ■

Theorem 4.11. *All J -sorted Hoehnke theories together with the corresponding theory morphisms form a category $Th_{dht}^\circ(J)$ and $sTh_{dht}^\circ(J)$, respectively, where the composition of theory morphisms is defined by the usual composition of functors.*

Proof. Obviously, $\text{dom}(F : \underline{\mathbf{T}} \rightarrow \underline{\mathbf{T}}') = \underline{\mathbf{T}}$, $\text{cod}(F : \underline{\mathbf{T}} \rightarrow \underline{\mathbf{T}}') = \underline{\mathbf{T}}'$.

The identical functor $1_{\underline{\mathbf{T}}} : \underline{\mathbf{T}} \rightarrow \underline{\mathbf{T}}$ is a theory morphism with respect to

$$\widetilde{1}_{\underline{\mathbf{T}}} = (\widetilde{1}_{\underline{\mathbf{T}}}\langle X, Y \rangle = 1_{X \otimes Y} \mid X, Y \in H^\circ), \quad i_{1_{\underline{\mathbf{T}}}} = 1_I.$$

Let $F : \underline{\mathbf{T}} \rightarrow \underline{\mathbf{T}}'$ and $G : \underline{\mathbf{T}}' \rightarrow \underline{\mathbf{T}}''$ be theory morphisms. Then, by definition, FG is a functor fulfilling the condition (Th1).

Moreover, because of Lemma 4.10 and Proposition 3.9, FG is a theory morphism.

Trivially, $F1_{\underline{\mathbf{T}}} = F = F1_{\underline{\mathbf{T}}'}$ and $F(GH) = (FG)H$ for every theory morphism F and all composable theory morphisms F , G and H . ■

Theorem 4.12. *Let $Th_{dht}^\circ(J)$ and $sTh_{dht}^\circ(J)$ be the categories introduced above. Then there are the functors*

$$\Sigma : Th_{dht}^\circ(J) \rightarrow sTh_{dht}^\circ(J)$$

$$\underline{\mathbf{T}} \mapsto \underline{\mathbf{T}}\Sigma := \underline{\mathbf{T}}_{\mathbf{s}} \quad (\text{see 4.6}),$$

$$(F : \underline{\mathbf{T}} \rightarrow \underline{\mathbf{T}}') \mapsto (F\Sigma : \underline{\mathbf{T}}_{\mathbf{s}} \rightarrow \underline{\mathbf{T}}'_{\mathbf{s}}) \quad \text{defined by}$$

$$XW^* \mapsto XW^*, \quad [\varphi]_{\mathcal{X}} \mapsto [\varphi F]_{\mathcal{X}'}$$

and

$$\Pi : sTh_{dht}^{\circ}(J) \rightarrow Th_{dht}^{\circ}(J)$$

$$\underline{\mathbf{T}}_s \mapsto \underline{\mathbf{T}}_s \Pi := \underline{\mathbf{T}} \text{ (see 4.7),}$$

$$(F : \underline{\mathbf{T}}_s \rightarrow \underline{\mathbf{T}}_s') \mapsto (F\Pi : \underline{\mathbf{T}} \rightarrow \underline{\mathbf{T}}') \text{ defined by}$$

$$X \mapsto X, (X, \varphi, Y) \mapsto (X, \varphi F, Y)$$

such that Σ is a left-adjoint functor of the functor Π .

Proof. a) The functor property of Σ :

The mapping on objects is well defined by Theorem 4.5. Let F be a theory morphism from a J -sorted theory $\underline{\mathbf{T}}$ into a J -sorted theory $\underline{\mathbf{T}}'$, i.e. $F \in Th_{dht}^{\circ}(J)[\underline{\mathbf{T}}, \underline{\mathbf{T}}']$. Then $F\Sigma$, defined as above, is a theory morphism too, more precisely,

$$F\Sigma \in sTh^{\circ}(J)[\underline{\mathbf{T}}\Sigma, \underline{\mathbf{T}}'\Sigma].$$

By definition, the mapping $F\Sigma$ respects “dom” and “cod” and one obtains

$$1_{XW^*}^{(\underline{\mathbf{T}}\Sigma)}(F\Sigma) = \left[1_X^{(\underline{\mathbf{T}})} \right]_{\varkappa} (F\Sigma) = \left[1_X^{(\underline{\mathbf{T}})} F \right]_{\varkappa} = \left[1_X^{(\underline{\mathbf{T}}')} \right]_{\varkappa} = 1_{XW^*}^{(\underline{\mathbf{T}}'\Sigma)} = 1_{(XW^*)(F\Sigma)}^{(\underline{\mathbf{T}}'\Sigma)}$$

for all objects $X \in |\underline{\mathbf{T}}|$.

Now let $[\varphi]_{\varkappa} \in \underline{\mathbf{T}}\Sigma[XW^*, YW^*]$, $[\psi]_{\varkappa} \in \underline{\mathbf{T}}\Sigma[UV^*, VW^*]$ be arbitrary morphisms such that $YW^* = UV^*$. Then

$$\begin{aligned} ([\varphi]_{\varkappa}[\psi]_{\varkappa})(F\Sigma) &= [\varphi c_{Y,U} \psi]_{\varkappa} (F\Sigma) = [\varphi F]_{\varkappa'} [c_{Y,U} F]_{\varkappa'} [\psi F]_{\varkappa'} \\ &= [\varphi F]_{\varkappa'} [c'_{Y,U}]_{\varkappa'} [\psi F]_{\varkappa'} = [\varphi F]_{\varkappa'} [1'_{YW^*, UV^*}]_{\varkappa'} [\psi F]_{\varkappa'} \\ &= [\varphi F]_{\varkappa'} [\psi F]_{\varkappa'} = [\varphi]_{\varkappa} (F\Sigma) [\psi]_{\varkappa} (F\Sigma). \end{aligned}$$

Furthermore, the functor $F\Sigma$ satisfies (Th1) by definition, (sFD) and (sFT) since for all $A \in S^{\circ}$ one has

$$d_A^{(\mathbf{T}\Sigma)}(F\Sigma) = \left[d_{AW}^{(\mathbf{T})} \right]_{\mathcal{X}}(F\Sigma) = \left[d_{AW}^{(\mathbf{T})} F \right]_{\mathcal{X}'} = \left[d_{(AW)F}^{(\mathbf{T}')} \right]_{\mathcal{X}'} = \left[d_{AW}^{(\mathbf{T}')} \right]_{\mathcal{X}'} = d_{A(F\Sigma)}^{(\mathbf{T}'\Sigma)}$$

and

$$t_A^{(\mathbf{T}\Sigma)}(F\Sigma) = \left[t_{AW}^{(\mathbf{T})} \right]_{\mathcal{X}}(F\Sigma) = \left[t_{AW}^{(\mathbf{T})} F \right]_{\mathcal{X}'} = \left[t_{(AW)F}^{(\mathbf{T}')} \right]_{\mathcal{X}'} = \left[t_{AW}^{(\mathbf{T}')} \right]_{\mathcal{X}'} = t_{A(F\Sigma)}^{(\mathbf{T}'\Sigma)},$$

and (sFM) since for all $\varphi \in \mathbf{T}[X, U]$, $\psi \in \mathbf{T}[Y, V]$ the equation

$$\begin{aligned} ([\varphi]_{\mathcal{X}} \otimes [\psi]_{\mathcal{X}})(F\Sigma) &= [(\varphi \otimes \psi)F]_{\mathcal{X}'} = [\varphi F \otimes \psi F]_{\mathcal{X}'} \\ &= [\varphi F]_{\mathcal{X}'} \otimes [\psi F]_{\mathcal{X}'} = [\varphi]_{\mathcal{X}}(F\Sigma) \otimes [\psi]_{\mathcal{X}}(F\Sigma) \end{aligned}$$

is valid.

b) The functor property of Π :

The mapping on objects $\underline{\mathbf{T}}_{\mathbf{s}}$ is well defined by Theorem 4.7.

Let $(F : \underline{\mathbf{T}}_{\mathbf{s}} \rightarrow \underline{\mathbf{T}}'_{\mathbf{s}})$ be a theory morphism. Then $(F\Pi : \underline{\mathbf{T}} \rightarrow \underline{\mathbf{T}}')$ defined by

$$X \mapsto X, (X, \varphi, Y) \mapsto (X, \varphi F, Y)$$

is a theory morphism too, since the conditions (Th1), (sFD), (sFT), and (sFM) are satisfied.

Ad (Th1): $\forall X \in H^\circ (X(F\Pi) = X)$ by definition.

Ad (sFD):

$$\begin{aligned} \forall X \in H^\circ \left(d_X^{(\mathbf{T})}(F\Pi) &= \left(X, d_{XW^*}^{(\mathbf{T}_{\mathbf{s}})}, X \otimes X \right) (F\Pi) = \left(X, d_{XW^*}^{(\mathbf{T}_{\mathbf{s}})} F, X \otimes X \right) \\ &= \left(X, d_{XW^*F}^{(\mathbf{T}'_{\mathbf{s}})}, X \otimes X \right) = \left(X, d_{XW^*}^{(\mathbf{T}'_{\mathbf{s}})}, X \otimes X \right) = d_X^{(\mathbf{T}')} = d_{X(F\Pi)}^{(\mathbf{T}')} \end{aligned}$$

$$\begin{aligned} \text{Ad (sFT): } \forall X \in H^\circ \left(t_X^{(\mathbf{T})}(F\Pi) &= \left(X, t_{XW^*}^{(\mathbf{T}_{\mathbf{s}})}, I \right) (F\Pi) = \left(X, t_{XW^*}^{(\mathbf{T}_{\mathbf{s}})} F, I \right) \\ &= \left(X, t_{XW^*F}^{(\mathbf{T}'_{\mathbf{s}})}, I \right) = \left(X, t_{XW^*}^{(\mathbf{T}'_{\mathbf{s}})}, I \right) = t_X^{(\mathbf{T}')} = t_{X(F\Pi)}^{(\mathbf{T}')} \end{aligned}$$

$$\begin{aligned}
\text{Ad (sFM): } \forall \rho \in \mathbf{T}[X, U], \sigma \in \mathbf{T}[Y, V] & \left((\rho \otimes \sigma)(F\Pi) \right. \\
& = ((X, \varphi, U) \otimes (Y, \psi, V))(F\Pi) \\
& = (X \otimes Y, \varphi \otimes \psi, U \otimes V)(F\Pi) \\
& = (X \otimes Y, (\varphi \otimes \psi)F, U \otimes V) \\
& = (X \otimes Y, \varphi F \otimes \psi F, U \otimes V) \\
& = (X, \varphi F, U) \otimes (Y, \psi F, V) \\
& = (X, \varphi, U)(F\Pi) \otimes (Y, \psi, V)(F\Pi) \\
& \left. = \rho(F\Pi) \otimes \sigma(F\Pi) \right).
\end{aligned}$$

c) It remains to show that Σ is a left-adjoint of Π . We will prove in several steps that for every $\underline{\mathbf{T}} \in |Th_{dht}^\circ(J)|$ and every $\underline{\mathbf{T}}_{\mathbf{s}} \in |sTh_{dht}^\circ(J)|$ there is an isomorphism between the sets $sTh_{dht}^\circ(J)[\underline{\mathbf{T}}\Sigma, \underline{\mathbf{T}}_{\mathbf{s}}]$ and $Th_{dht}^\circ(J)[\underline{\mathbf{T}}, \underline{\mathbf{T}}_{\mathbf{s}}\Pi]$.

1. A functor from a theory $\underline{\mathbf{T}}$ into $\underline{\mathbf{T}}(\Sigma\Pi)$:

Define a mapping $\Theta_{\underline{\mathbf{T}}}$ on objects and morphisms of any Hoehnke theory by $X\Theta_{\underline{\mathbf{T}}} := X$ and $\varphi\Theta_{\underline{\mathbf{T}}} := (X, [\varphi]_{\varkappa}, Y)$ for $\varphi \in \underline{\mathbf{T}}[X, Y]$. This mappings are well defined and the values are objects and morphisms of $\underline{\mathbf{T}}(\Sigma\Pi)$.

$\Theta_{\underline{\mathbf{T}}} : \underline{\mathbf{T}} \rightarrow \underline{\mathbf{T}}(\Sigma\Pi)$ is a functor, since the object mapping is compatible with “dom” and “cod” and

$$\begin{aligned}
1_X^{(\underline{\mathbf{T}})}\Theta_{\underline{\mathbf{T}}} &= \left(X, \left[1_X^{(\underline{\mathbf{T}})} \right]_{\varkappa}, X \right) = \left(X, 1_{XW^*}^{(\underline{\mathbf{T}}(\Sigma))}, X \right) = 1_X^{((\underline{\mathbf{T}}\Sigma)\Pi)} = 1_{X\Theta_{\underline{\mathbf{T}}}}^{(\underline{\mathbf{T}}(\Sigma\Pi))}, \\
(\varphi\psi)\Theta_{\underline{\mathbf{T}}} &= (X, [\varphi\psi]_{\varkappa}, U) = (X, [\varphi]_{\varkappa}[\psi]_{\varkappa}, U) \\
&= (X, [\varphi]_{\varkappa}, Y)(Y, [\psi]_{\varkappa}, U) = (\varphi\Theta_{\underline{\mathbf{T}}})(\psi\Theta_{\underline{\mathbf{T}}}).
\end{aligned}$$

Moreover, $\Theta_{\mathbf{T}} : \underline{\mathbf{T}} \rightarrow \underline{\mathbf{T}}(\Sigma\Pi)$ is even a theory morphism because of the validity of (Th1), (sFD), (sFT), and (sFM) as follows:

$$\forall X \in |\mathbf{T}| \quad (X\Theta_{\mathbf{T}} = X) \text{ by definition.}$$

$$\begin{aligned} \forall X \in |\mathbf{T}| \quad \left(d_X^{(\mathbf{T})}\Theta_{\mathbf{T}} = \left(X, \left[d_X^{(\mathbf{T})} \right]_{\varkappa}, X \otimes X \right) &= \left(X, d_{XW^*}^{(\mathbf{T}\Sigma)}, X \otimes X \right) \\ &= d_X^{((\mathbf{T}\Sigma)\Pi)} = d_{X\Theta_{\mathbf{T}}}^{(\mathbf{T}(\Sigma\Pi))}. \end{aligned}$$

$$\begin{aligned} \forall X \in |\mathbf{T}| \quad \left(t_X^{(\mathbf{T})}\Theta_{\mathbf{T}} = \left(X, \left[t_X^{(\mathbf{T})} \right]_{\varkappa}, I \right) &= \left(X, t_{XW^*}^{(\mathbf{T}\Sigma)}, I \right) = t_X^{((\mathbf{T}\Sigma)\Pi)} \\ &= t_{X\Theta_{\mathbf{T}}}^{(\mathbf{T}(\Sigma\Pi))}. \end{aligned}$$

$$\begin{aligned} \forall \varphi \in \mathbf{T}[X, U], \quad \psi \in \mathbf{T}[Y, V] \quad ((\varphi \otimes \psi)\Theta_{\mathbf{T}} &= (X \otimes Y, [\varphi \otimes \psi]_{\varkappa}, U \otimes V) \\ &= (X, [\varphi]_{\varkappa}, U) \otimes (Y, [\psi]_{\varkappa}, V) = \varphi\Theta_{\mathbf{T}} \otimes \psi\Theta_{\mathbf{T}}. \end{aligned}$$

In such a way, every theory morphism $G' \in |sTh_{dht}^{\circ}(J)|$ determines uniquely a theory morphism $G := \Theta_T(G'\Pi) \in Th_{dht}^{\circ}(J)[\underline{\mathbf{T}}, \underline{\mathbf{T}}_s\Pi]$.

2. A construction of a strictly d -monoidal functor $\overline{G} : \underline{\mathbf{T}} \rightarrow \underline{\mathbf{T}}_s$:

To every theory morphism $G \in Th_{dht}^{\circ}(J)[\underline{\mathbf{T}}, \underline{\mathbf{T}}_s\Pi]$ there is assigned in a natural manner a strictly d -monoidal functor $\overline{G} : \underline{\mathbf{T}} \rightarrow \underline{\mathbf{T}}_s$ as follows:

Let be given any $G \in Th_{dht}^{\circ}(J)[\underline{\mathbf{T}}, \underline{\mathbf{T}}_s\Pi]$. Then

$$XG = X \quad (X \in |\mathbf{T}|) \text{ and}$$

$$\mathbf{T}[X, U] \ni \varphi \mapsto \varphi G = (X, \varphi_G, U) \in \mathbf{T}_s\Pi[X, U],$$

$$\text{where } \varphi_G \in \mathbf{T}_s[XW^*, UW^*].$$

The agreements

$$H^{\circ} \ni X \mapsto X\Xi := XW^* \in S^{\circ}$$

and

$$\mathbf{T}_s\Pi[X, U] \ni (X, \psi, U) \mapsto (X, \psi, U)\Xi := \psi \in \mathbf{T}_s[XW^*, UW^*]$$

define a functor $\Xi : \mathbf{T}_s\Pi \rightarrow \mathbf{T}_s$ because of:

$$\text{dom}^{(\mathbf{T}_s)}((X, \psi, U)\Xi) = \text{dom}^{(\mathbf{T}_s)}(\psi) = XW^* = X\Xi = \left(\text{dom}^{(\mathbf{T}_s\Pi)}(X, \psi, U) \right) \Xi,$$

$$\text{cod}^{(\mathbf{T}_s)}((X, \psi, U)\Xi) = \text{cod}^{(\mathbf{T}_s)}(\psi) = UW^* = U\Xi = \left(\text{cod}^{(\mathbf{T}_s\Pi)}(X, \psi, U) \right) \Xi,$$

$$1_X^{(\mathbf{T}_s\Pi)}\Xi = \left(X, 1_{XW^*}^{(\mathbf{T}_s)}, X \right) \Xi = 1_{XW^*}^{(\mathbf{T}_s)} = 1_{X\Xi}^{(\mathbf{T}_s)},$$

$$((X, \psi_1, U)(U, \psi_2, Y))\Xi = (X, \psi_1\psi_2, Y)\Xi = \psi_1\psi_2 = (X, \psi_1, U)\Xi(U, \psi_2, Y)\Xi.$$

$\Xi : \mathbf{T}_s\Pi \rightarrow \mathbf{T}_s$ is a strictly d -monoidal functor since (sFD), (sFT), and (sFM) are valid:

$$d_X^{(\mathbf{T}_s\Pi)}\Xi = \left(X, d_{XW^*}^{(\mathbf{T}_s)}, X \otimes X \right) \Xi = d_{XW^*}^{(\mathbf{T}_s)} = d_{X\Xi}^{(\mathbf{T}_s)},$$

$$t_X^{(\mathbf{T}_s\Pi)}\Xi = \left(X, t_{XW^*}^{(\mathbf{T}_s)}, I \right) \Xi = t_{XW^*}^{(\mathbf{T}_s)} = t_{X\Xi}^{(\mathbf{T}_s)},$$

$$((X_1, \psi_1, U_1) \otimes (X_2, \psi_2, U_2))\Xi = (X_1 \otimes X_2, \psi_1 \otimes \psi_2, U_1 \otimes U_2)\Xi$$

$$= \psi_1 \otimes \psi_2 = (X_1, \psi_1, U_1)\Xi \otimes (X_2, \psi_2, U_2)\Xi.$$

The compositum $\overline{G} := G\Xi$ is strictly d -monoidal functor from \mathbf{T} into \mathbf{T}_s .

3. The induced theory morphism $G' \in sTh_{dht}^\circ(J)$:

Let G , Ξ , and \overline{G} be given as above. Then define a mapping G' by $AG' := A$ for all $A \in S^\circ$ and $[\varphi]_{\varkappa}G' := \varphi\overline{G} = (\varphi G)\Xi = (X, \varphi_G, U)\Xi = \varphi_G \in \mathbf{T}_s[XW^*, UW^*]$ for all $\varphi \in \mathbf{T}[X, U]$, where φ_G is a well-defined morphism of \mathbf{T}_s .

Because of

$$\varphi_1 \in \mathbf{T}[X_1, U_1] \wedge \varphi_2 \in \mathbf{T}[X_2, U_2] \wedge [\varphi_1]_{\varkappa} = [\varphi_2]_{\varkappa} \Rightarrow$$

$$\Rightarrow X_1W^* = X_2W^* := A \wedge U_1W^* = U_2W^* := B$$

$$\wedge c_{X_1}^{-1}\varphi_1c_{U_1} = c_{X_2}^{-1}\varphi_2c_{U_2} \in \mathbf{T}_s[AW, BW] \Rightarrow$$

$$\begin{aligned}
&\Rightarrow (c_{X_1}^{-1}G)(\varphi_1G)(c_{U_1}G) = (c_{X_2}^{-1}G)(\varphi_2G)(c_{U_2}G) \in \mathbf{T}_s\Pi[A, B] \\
&\Rightarrow \left(AW, 1_A^{(\mathbf{T}_s)}, X_1\right) (X_1, (\varphi_1)_G, U_1) \left(U_1, 1_B^{(\mathbf{T}_s)}, BW\right) \\
&= \left(AW, 1_A^{(\mathbf{T}_s)}, X_2\right) (X_2, (\varphi_2)_G, U_2) \left(U_2, 1_B^{(\mathbf{T}_s)}, BW\right) \\
&\Rightarrow (X_1, (\varphi_1)_G, U_1) = (X_2, (\varphi_2)_G, U_2) \\
&\Rightarrow (\varphi_1)_G = (\varphi_2)_G,
\end{aligned}$$

possibly different representants of the same \varkappa -class of morphisms determine identical images, thus $[\varphi_1]_{\varkappa}G' = [\varphi_2]_{\varkappa}G'$.

The mapping G' determines a functor $G' : \underline{\mathbf{T}}\Sigma \rightarrow \underline{\mathbf{T}}_s$ since

$$\text{dom}^{(\mathbf{T}_s)}([\varphi]_{\varkappa}G') = \text{dom}^{(\mathbf{T}_s)}(\varphi_G) = XW^* = (XW^*)G' = \left(\text{dom}^{(\mathbf{T}\Sigma)}([\varphi]_{\varkappa})\right) G',$$

$$\text{cod}^{(\mathbf{T}_s)}([\varphi]_{\varkappa}G') = \text{cod}^{(\mathbf{T}_s)}(\varphi_G) = UW^* = (UW^*)G' = \left(\text{cod}^{(\mathbf{T}\Sigma)}([\varphi]_{\varkappa})\right) G',$$

$$\begin{aligned}
\left(1_A^{(\mathbf{T}\Sigma)}\right) G' &= \left(\left[1_{AW}^{(\mathbf{T})}\right]_{\varkappa}\right) G' = \left(1_{AW}^{(\mathbf{T})}\right) \overline{G} = \left(1_{AW}^{(\mathbf{T})}\right) (G\Xi) = \left(\left(1_{AW}^{(\mathbf{T})}\right) G\right) \Xi \\
&= \left(1_{(AW)G}^{(\mathbf{T}_s\Pi)}\right) \Xi = 1_{((AW)G)\Xi}^{(\mathbf{T}_s)} = 1_A^{(\mathbf{T}_s)} = 1_{AG'}^{(\mathbf{T}_s)},
\end{aligned}$$

$$([\varphi]_{\varkappa}[\psi]_{\varkappa})G' = ([\varphi c_{U,Y}\psi]_{\varkappa})G' = (\varphi c_{U,Y}\psi)\overline{G} = (\varphi\overline{G})(c_{U,Y}\overline{G})(\psi\overline{G})$$

$$= \varphi_G\psi_G = ([\varphi]_{\varkappa}G')([\psi]_{\varkappa}G').$$

Moreover, G' is even a theory morphism in $sTh_{dht}^{\circ}(J)$ because of the validity of (Th1) by definition and the validity of (sFD), (sFT), and (sFM) as follows:

$$\left(d_A^{(\mathbf{T}\Sigma)}\right) G' = \left[d_A^{(\mathbf{T})}\right]_{\varkappa} G' = d_A^{(\mathbf{T})}\overline{G} = d_{AW}^{(\mathbf{T}_s\Pi)}\Xi = d_A^{(\mathbf{T}_s)} = d_{AG'}^{(\mathbf{T}_s)},$$

$$\left(t_A^{(\mathbf{T}\Sigma)}\right) G' = \left[t_A^{(\mathbf{T})}\right]_{\varkappa} G' = t_A^{(\mathbf{T})}\overline{G} = t_{AW}^{(\mathbf{T}_s\Pi)}\Xi = t_A^{(\mathbf{T}_s)} = t_{AG'}^{(\mathbf{T}_s)},$$

$$\begin{aligned}
([\varphi]_{\varkappa} \otimes [\psi]_{\varkappa})G' &= ([\varphi \otimes \psi]_{\varkappa})G' = (\varphi \otimes \psi)\overline{G} = (\varphi\overline{G}) \otimes (\psi\overline{G}) = \\
&= [\varphi]_{\varkappa}G' \otimes [\psi]_{\varkappa}G'.
\end{aligned}$$

By the functor $\Pi : sTh^{\circ} - dht(J) \rightarrow Th_{dht}^{\circ}(J)$, $G'\Pi : \mathbf{T}(\Sigma\Pi) \rightarrow \mathbf{T}_s\Pi$ is a theory morphism.

Moreover, this theory morphism has the property

$$G = \Theta_{\mathbf{T}}(G'\Pi).$$

This is a consequence of

$$H^{\circ} \ni X \mapsto X(\Theta_{\mathbf{T}}(G'\Pi)) = (X\Theta_{\mathbf{T}})(G'\Pi) = X(G'\Pi) = X = XG$$

and

$$\begin{aligned}
\mathbf{T}[X, U] \ni \varphi \mapsto \varphi(\Theta_{\mathbf{T}}(G'\Pi)) &= (\varphi\Theta_{\mathbf{T}})(G'\Pi) = (X, [\varphi]_{\varkappa}, U)(G'\Pi) = \\
&= (X, [\varphi]_{\varkappa}G', U) = (X, \varphi_G, U) = \varphi G.
\end{aligned}$$

Finally, let $L : \mathbf{T}\Sigma \rightarrow \mathbf{T}_s$ be a theory morphism such that $\Theta_{\mathbf{T}}(L\Pi) = G$. Then

$$\forall X \in H^{\circ} ((XW^*)G' = XW^* = (XW^*)G)$$

and

$$\begin{aligned}
\forall X, U \in H^{\circ} \forall \varphi \in \mathbf{T}[X, U] ((X, [\varphi]_{\varkappa}G', U) &= (X, \varphi\overline{G}, U) = \varphi G = \\
&= \varphi(\Theta_{\mathbf{T}}(L\Pi)) = (\varphi\Theta_{\mathbf{T}})(L\Pi) = (X, [\varphi]_{\varkappa}, U)(L\Pi) = (X, [\varphi]_{\varkappa}L, U) \\
&\Rightarrow [\varphi]_{\varkappa}G' = [\varphi]_{\varkappa}L),
\end{aligned}$$

thus $L = G'$, i.e. G' is the only theory morphism in $sTh_{dht}^{\circ}(J)$ with the property

$$G = \Theta_{\mathbf{T}}(G'\Pi). \quad \blacksquare$$

$$\begin{array}{ccc}
\underline{\mathbf{T}}(\Sigma\Pi) & & \\
\uparrow \Theta_{\mathbf{T}} & \searrow G'\Pi & \\
\underline{\mathbf{T}} & \xrightarrow{G} & \underline{\mathbf{T}}_s\Pi \\
\downarrow \Psi & \searrow \bar{G} & \downarrow \Xi \\
\underline{\mathbf{T}}\Sigma & \xrightarrow{G'} & \underline{\mathbf{T}}_s
\end{array}$$

The diagram illustrates the individual d -monoidal functors and theory morphisms, respectively, which are considered in the proof of the last theorem. This diagram is commutative in all of its parts, namely $G = \Theta_{\mathbf{T}}(G'\Pi)$ was shown above, $\bar{G} = G\Xi$ by definition, and $\bar{G} = \Psi G'$ follows by

$$X(\Psi G') = (X\Psi)G' = (XW^*)G' = XW^* = X\bar{G}$$

and

$$\varphi(\Psi G') = (\varphi\Psi)G' = [\varphi]_{\neq}G' = \varphi_G = \varphi\bar{G}.$$

Corollary 4.13. *The theory morphisms $\Theta_{\mathbf{T}}$, $\underline{\mathbf{T}} \in |\mathit{Th}_{dht}^{\circ}(J)|$ form a natural transformation $\Theta : \mathit{Id}_{\mathit{Th}_{dht}^{\circ}(J)} \rightarrow \Sigma\Pi$.*

Proof. $\Theta = (\Theta_{\mathbf{T}} \mid \underline{\mathbf{T}} \in |\mathit{Th}_{dht}^{\circ}(J)|)$ is a natural transformation $\Theta : \mathit{Id}_{\mathit{Th}_{dht}^{\circ}(J)} \rightarrow \Sigma\Pi$ because of the commutativity of the following diagram for arbitrary theories and theory morphisms of $\mathit{Th}_{dht}^{\circ}(J)$:

$$\begin{array}{ccc}
(\mathbf{N}) & \mathbf{T} & \xrightarrow{\Theta_{\mathbf{T}}} & \mathbf{T}(\Sigma\Pi) \\
& \downarrow F & & \downarrow F(\Sigma\Pi) \\
& \mathbf{T}' & \xrightarrow{\Theta_{\mathbf{T}'}} & \mathbf{T}'(\Sigma\Pi)
\end{array}$$

Let X be any object of $\underline{\mathbf{T}}$. Then

$$X(F\Theta_{\mathbf{T}'}) = (XF)\Theta_{\mathbf{T}'} = X\Theta_{\mathbf{T}'} = X$$

and

$$X(\Theta_{\mathbf{T}}F(\Sigma\Pi)) = (X\Theta_{\mathbf{T}})((F\Sigma)\Pi) = X.$$

For every morphism $\varphi \in \mathbf{T}[X, U]$ one has

$$\varphi(F\Theta_{\mathbf{T}'}) = (\varphi F)\Theta_{\mathbf{T}'} = (X, [\varphi F]_{\mathcal{A}'}, U)$$

and

$$\begin{aligned}
\varphi(\Theta_{\mathbf{T}}F(\Sigma\Pi)) &= (\varphi\Theta_{\mathbf{T}})((F\Sigma)\Pi) = \\
&= (X, [\varphi]_{\mathcal{A}}, U)((F\Sigma)\Pi) = \\
&= (X, [\varphi]_{\mathcal{A}}(F\Sigma), U) = (X, [\varphi F]_{\mathcal{A}'}, U),
\end{aligned}$$

hence

$$\Theta_{\mathbf{T}}F(\Sigma\Pi) = F\Theta_{\mathbf{T}'}. \quad \blacksquare$$

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