

**RANK AND PERIMETER PRESERVER
OF RANK-1 MATRICES OVER MAX ALGEBRA**

SEOK-ZUN SONG

AND

KYUNG-TAE KANG

Department of Mathematics, Cheju National University
Jeju 690-756, Republic of Korea

e-mail: szsong@cheju.ac.kr

e-mail: kangkt@cheju.ac.kr

Abstract

For a rank-1 matrix $A = \mathbf{a} \otimes \mathbf{b}^t$ over max algebra, we define the *perimeter* of A as the number of nonzero entries in both \mathbf{a} and \mathbf{b} . We characterize the linear operators which preserve the rank and perimeter of rank-1 matrices over max algebra. That is, a linear operator T preserves the rank and perimeter of rank-1 matrices if and only if it has the form $T(A) = U \otimes A \otimes V$, or $T(A) = U \otimes A^t \otimes V$ with some monomial matrices U and V .

Keywords: max algebra; semiring; linear operator; monomial; rank; dominate; perimeter; (U, V) -operator.

2000 Mathematics Subject Classification: 15A03, 15A04, 12K10, 16Y60.

1. INTRODUCTION AND PRELIMINARIES

There are many papers on the study of linear operators that preserve rank of matrices over several semirings. Beasley and Pullman ([3]) obtained characterizations of rank-preserving operators of Boolean matrices. Bapat, Pati and Song ([2]) obtained characterizations of linear operators that preserve the rank of matrices over max algebra. They did not find necessary and sufficient conditions for an operator to preserve the rank of rank-1 matrices over max algebra. We consider characterizations of the linear operator that preserve the rank of the rank-1 matrices over max algebra.

The *max algebra* consists of the set \mathbb{R}_{\max} , where \mathbb{R}_{\max} is the set of non-negative real numbers, equipped with two binary operations, which we call addition (\oplus) and multiplication (\cdot). The operations are defined as $a \oplus b = \max\{a, b\}$ and $a \cdot b = ab$. That is, their sum is the maximum of a and b and their product is the usual product in the reals. There has been a great deal of interest in recent years in this max algebra. This system allows us to express some nonlinear phenomena in the conventional algebra in a linear fashion ([1]).

Let $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ denote the set of all $m \times n$ matrices with entries in \mathbb{R}_{\max} . The (i, j) th entry of a matrix A is denoted by a_{ij} . If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices over \mathbb{R}_{\max} , then the sum of A and B is denoted by $A \oplus B$, which is the $m \times n$ matrix with $a_{ij} \oplus b_{ij}$ as its (i, j) th entry. If $c \in \mathbb{R}_{\max}$, then cA is the matrix $[ca_{ij}]$. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then their product is denoted by $A \otimes B$, which is the $m \times p$ matrix with $\max\{a_{ir}b_{rj} | r = 1, \dots, n\}$ as its (i, j) th entry. The zero matrix is denoted by O . The identity matrix of an appropriate order is denoted by I . And the transpose of $A = [a_{ij}]$, denoted by A^t , is defined in the usual way. That is, the (i, j) th entry of A^t is a_{ji} for all i and j . Throughout this paper, we shall adopt the convention that $m \leq n$ unless otherwise specified.

The *rank* or *factor rank*, $r(A)$, of a nonzero matrix $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$ is defined as the least integer k for which there exist $m \times k$ and $k \times n$ matrices B and C with $A = B \otimes C$. The rank of a zero matrix is zero. It is well known that $r(A)$ is the least k such that A is the sum of k matrices of rank 1 (see [5], [4]).

Let $\Delta_{m,n} = \{(i, j) | 1 \leq i \leq m, 1 \leq j \leq n\}$, and E_{ij} be the $m \times n$ matrix whose (i, j) th entry is 1 and whose other entries are all 0, and $\mathbb{E}_{m,n} = \{E_{ij} | (i, j) \in \Delta_{m,n}\}$. We call E_{ij} a *cell*.

A square matrix A over \mathbb{R}_{\max} is called *monomial* if it has exactly one nonzero element in each row and column. Since $\mathcal{M}_{n,n}(\mathbb{R}_{\max})$ is a semiring, we can consider the invertible matrices under multiplication. The monomial matrices are precisely invertible matrices over \mathbb{R}_{\max} (see [2]).

If A and B are in $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$, we say A *dominates* B (written $B \leq A$ or $A \geq B$) if $a_{ij} = 0$ implies $b_{ij} = 0$ for all i, j .

For example, if

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix},$$

then we have $A \leq B$ and $B \leq A$, but $A \neq B$.

Also lowercase, boldface letters will represent column vectors, all vectors \mathbf{u} are column vectors (\mathbf{u}^t is a row vector) for $\mathbf{u} \in \mathbb{R}_{\max}^m [= \mathcal{M}_{m,1}(\mathbb{R}_{\max})]$.

It is easy to verify that the rank of $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$ is 1 if and only if there exist nonzero vectors $\mathbf{a} \in \mathcal{M}_{m,1}(\mathbb{R}_{\max})$ and $\mathbf{b} \in \mathcal{M}_{n,1}(\mathbb{R}_{\max})$ such that $A = \mathbf{a} \otimes \mathbf{b}^t$. We call \mathbf{a} the *left factor*, and \mathbf{b} the *right factor* of A . But these vectors \mathbf{a} and \mathbf{b} are not uniquely determined by A .

For example,

$$\begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \dots$$

For any vector $\mathbf{u} \in \mathcal{M}_{m,1}(\mathbb{R}_{\max})$, we define $|\mathbf{u}|$ to be the number of nonzero entries in \mathbf{u} . Let $A = [a_{ij}]$ be any matrix in $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$. Then we define $A^* = [a_{ij}^*]$ to be the $m \times n$ $(0,1)$ -matrix whose (i,j) th entry is 1 if and only if $a_{ij} \neq 0$ for $A = [a_{ij}]$.

It follows from the definition that

$$(1.1) \quad (A \otimes B)^* = A^* \otimes B^* \quad \text{and} \quad (B \oplus C)^* = B^* \oplus C^*$$

for all $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$ and all $B, C \in \mathcal{M}_{n,r}(\mathbb{R}_{\max})$. Also we can easily obtain that $A \geq B$ if and only if $A \oplus B = A$ for any $m \times n$ $(0,1)$ -matrix A and B .

Lemma 1.1. *For any factorization $\mathbf{a} \otimes \mathbf{b}^t$ of an $m \times n$ rank-1 matrix A , $|\mathbf{a}|$ and $|\mathbf{b}|$ are uniquely determined by A .*

Proof. Consider the $m \times n$ $(0,1)$ -matrix $A^* = [a_{ij}^*]$ whose (i,j) th entry is 1 if and only if $a_{ij} \neq 0$. By (1.1), $A^* = \mathbf{a}^* \otimes (\mathbf{b}^*)^t$ is the rank-1 matrix. Since A^* is the $(0,1)$ matrix, it is easy to show that $|\mathbf{a}^*|$ and $|\mathbf{b}^*|$ are uniquely determined by A^* . Therefore, $|\mathbf{a}|$ and $|\mathbf{b}|$ are uniquely determined by A . ■

Let A be any rank-1 matrix in $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$. We define the *perimeter* of A , $P(A)$, as $|\mathbf{a}| + |\mathbf{b}|$ for arbitrary factorization $A = \mathbf{a} \otimes \mathbf{b}^t$. Even though the factorizations of A are not unique, Lemma 1.1 shows that the perimeter of A is unique, and that $P(A) = P(A^*)$.

Proposition 1.2. *If A, B and $A \oplus B$ are rank-1 matrices in $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$, then $P(A \oplus B) < P(A) + P(B)$.*

Proof. Let $A = \mathbf{a} \otimes \mathbf{x}^t, B = \mathbf{b} \otimes \mathbf{y}^t$ and $A \oplus B = \mathbf{c} \otimes \mathbf{z}^t$ be any factorizations of A, B and $A \oplus B$. Then we have for all i, j

$$(1.2) \quad a_i \mathbf{x} \oplus b_j \mathbf{y} = c_j \mathbf{z}$$

and

$$(1.3) \quad x_j \mathbf{a} \oplus y_j \mathbf{b} = z_j \mathbf{c}.$$

If $B \leq A$, we have $(A \oplus B)^* = A^* \oplus B^* = A^*$. Thus we obtain that

$$P(A \oplus B) = P((A \oplus B)^*) = P(A^*) = P(A) < P(A) + P(B)$$

because $P(B) \neq 0$, as required.

The similar argument shows that if $A \leq B$, then $P(A \oplus B) < P(A) + P(B)$. So we can assume that $A \not\leq B$ and $B \not\leq A$. We consider the three cases.

Case 1. $\mathbf{a} \not\leq \mathbf{b}$ and $\mathbf{b} \not\leq \mathbf{a}$. The equation (1.2) implies that $a_i \mathbf{x} = c_j \mathbf{z}$ and $b_j \mathbf{y} = c_j \mathbf{z}$ for some nonzero scalars $a_i, c_i, b_j, c_j \in \mathbb{R}_{\max}$. Thus we have the following

$$\begin{aligned} P(A \oplus B) &= P\left(\left(\frac{c_i}{a_i} \mathbf{a} \oplus \frac{c_j}{b_j} \mathbf{b}\right) \otimes \mathbf{z}^t\right) = \left|\frac{c_i}{a_i} \mathbf{a} \oplus \frac{c_j}{b_j} \mathbf{b}\right| + |\mathbf{z}| \\ &< (|\mathbf{a}| + |\mathbf{z}|) + (|\mathbf{b}| + |\mathbf{z}|) \\ &= (|\mathbf{a}| + |\mathbf{x}|) + (|\mathbf{b}| + |\mathbf{y}|) = P(A) + P(B), \end{aligned}$$

as required.

Case 2. $\mathbf{a} \leq \mathbf{b}$. Then $\mathbf{x} \not\leq \mathbf{y}$ (and so $\mathbf{x}^* \not\leq \mathbf{y}^*$). Also the equation (1.3) becomes

$$(1.4) \quad x_j^* \mathbf{a}^* \oplus y_j^* \mathbf{b}^* = z_j^* \mathbf{c}^*.$$

Thus we have $x_j^* \mathbf{a}^* = z_j^* \mathbf{c}^*$ for some nonzero scalars $x_j, z_j \in \mathbb{R}_{\max}$. Since $x_j^* = z_j^* = 1$, we have $\mathbf{a}^* = \mathbf{c}^*$. But $\mathbf{b}^* \leq \mathbf{c}^*$ from (1.4). Therefore, $\mathbf{a}^* = \mathbf{b}^* = \mathbf{c}^*$ and we have

$$\begin{aligned}
P(A \oplus B) &= P((A \oplus B)^*) = P(\mathbf{c}^* \otimes (\mathbf{x}^* \oplus \mathbf{y}^*)^t) = |\mathbf{c}^*| + |\mathbf{x}^* \oplus \mathbf{y}^*| \\
&< (|\mathbf{c}^*| + |\mathbf{x}^*|) + (|\mathbf{c}^*| + |\mathbf{y}^*|) = (|\mathbf{a}^*| + |\mathbf{x}^*|) + (|\mathbf{b}^*| + |\mathbf{y}^*|) \\
&= (|\mathbf{a}| + |\mathbf{x}|) + (|\mathbf{b}| + |\mathbf{y}|) = P(A) + P(B),
\end{aligned}$$

as required.

Case 3. $\mathbf{b} \leq \mathbf{a}$. It is similar to the Case 2. ■

A mapping $T : \mathcal{M}_{m,n}(\mathbb{R}_{\max}) \rightarrow \mathcal{M}_{m,n}(\mathbb{R}_{\max})$ is called a *linear operator* if T has the following two properties:

- (1) $T(0) = 0$ and
- (2) $T(\alpha A \oplus \beta B) = \alpha T(A) \oplus \beta T(B)$ for all $A, B \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$ and for all $\alpha, \beta \in \mathbb{R}_{\max}$.

A linear operator $T : \mathcal{M}_{m,n}(\mathbb{R}_{\max}) \rightarrow \mathcal{M}_{m,n}(\mathbb{R}_{\max})$ is *invertible* if T is injective and surjective. As with vector space over fields, the inverse, T^{-1} , of a linear operator T is also linear.

Bapat, Pati and Song obtain the following:

Lemma 1.3. ([2]) *If T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$, then T is invertible if and only if T permutes $\mathbb{E}_{m,n}$ with some nonzero scalar multiplication.* ■

In this paper, we characterize the linear operators that preserve the rank and the perimeter of every rank-1 matrix over max algebra. These are motivated by analogous results for the linear operator which preserves all ranks in $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$. However, we obtain results and proofs in the view of the perimeter analog.

2. RANK AND PERIMETER PRESERVERS OF RANK-1 MATRICES OVER MAX ALGEBRA

In this section, we will characterize the linear operators that preserve the rank and the perimeter of every rank-1 matrix in $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$.

Suppose T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$. Then:

- (1) T is a (U, V) -operator if there exist monomials $U \in \mathcal{M}_{m,m}(\mathbb{R}_{\max})$ and $V \in \mathcal{M}_{n,n}(\mathbb{R}_{\max})$ such that $T(A) = U \otimes A \otimes V$ for all A in $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$, or $m = n$ and $T(A) = U \otimes A^t \otimes V$ for all A in $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$;
- (2) T preserve rank 1 if $r(T(A)) = 1$ whenever $r(A) = 1$ for all $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$;
- (3) T preserve perimeter k of rank-1 matrices if $P(T(A)) = k$ whenever $P(A) = k$ for all $A \in \mathcal{M}_{m,n}(\mathbb{R}_{\max})$ with $r(A) = 1$.

Proposition 2.1. *If T is a (U, V) -operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$, then T preserves both rank and perimeter of rank-1 matrices.*

Proof. Since T is a (U, V) -operator, there exist monomials $U \in \mathcal{M}_{m,m}(\mathbb{R}_{\max})$ and $V \in \mathcal{M}_{n,n}(\mathbb{R}_{\max})$ such that either $T(A) = U \otimes A \otimes V$ or $m = n$, $T(A) = U \otimes A^t \otimes V$ for all A in $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$. Let A be a matrix in $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ with $r(A) = 1$ and $A = \mathbf{a} \otimes \mathbf{b}^t$ be any factorization of A with $P(A) = |\mathbf{a}| + |\mathbf{b}|$. For the case $T(A) = U \otimes A \otimes V$,

$$T(A) = U \otimes A \otimes V = (U \otimes \mathbf{a}) \otimes (\mathbf{b}^t \otimes V) = (U \otimes \mathbf{a}) \otimes (V^t \otimes \mathbf{b})^t.$$

Thus we have

$$r(T(A)) = r((U \otimes \mathbf{a}) \otimes (V^t \otimes \mathbf{b})^t) = 1,$$

and

$$P(T(A)) = |U \otimes \mathbf{a}| + |V^t \otimes \mathbf{b}| = |\mathbf{a}| + |\mathbf{b}| = P(A).$$

For the case $T(A) = U \otimes A^t \otimes V$, we can show that $r(T(A)) = 1$ and $P(T(A)) = |\mathbf{a}| + |\mathbf{b}|$ by the similar method as above.

Hence (U, V) -operator preserves rank and perimeter of every rank-1 matrix. ■

We note that an $m \times n$ matrix has perimeter 2 if and only if it is a cell with nonzero scalar multiplication. Thus, we have the following Lemma:

Lemma 2.2. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$. If T preserves rank 1 and perimeter 2 of rank-1 matrices, then the following statements hold:*

- (1) T maps a cell into a cell with nonzero scalar multiplication;
- (2) T maps a row (or a column) of a matrix into a row or a column (if $m = n$) with scalar multiplication.

Proof. (1): It follows from the property that T preserves perimeter 2.

(2): If not, then there exists two distinct cells E_{ij}, E_{ih} in some i -th row such that $T(E_{ij})$ and $T(E_{ih})$ lie in two different rows and different columns. Then the rank of $E_{ij} \oplus E_{ih}$ is 1 but that of $T(E_{ij} \oplus E_{ih}) = T(E_{ij}) \oplus T(E_{ih})$ is 2. Therefore, T does not preserve rank 1, a contradiction. ■

The following is an example of a linear operator that preserves rank 1 and perimeter 2 of rank-1 matrices, but the operator does not preserve perimeter 3 and is not a (U, V) -operator.

Example 2.3. Let $T : \mathcal{M}_{2,2}(\mathbb{R}_{\max}) \rightarrow \mathcal{M}_{2,2}(\mathbb{R}_{\max})$ be defined by

$$T \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a \oplus b \oplus c \oplus d) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then it is easy to verify that T is a linear operator and preserve rank 1 and perimeter 2. But T does not preserve perimeter 3: For, $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \otimes [1 \ 0]$ has rank 1 and perimeter 3: but $T(A) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \otimes [1 \ 0]$ has rank 1 and perimeter 2. Moreover, T is not a (U, V) -operator: For, let $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in \mathcal{M}_{2,2}(\mathbb{R}_{\max})$. Then $T(X) = T \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}$. So, we cannot find monomials $U, V \in \mathcal{M}_{2,2}(\mathbb{R}_{\max})$ such that $T(X) = U \otimes X \otimes V$. This shows that T is not a (U, V) -operator.

Let $R_i = \{E_{ij} | 1 \leq j \leq n\}$, $C_j = \{E_{ij} | 1 \leq i \leq m\}$, $\mathcal{R} = \{R_i | 1 \leq i \leq m\}$ and $\mathcal{C} = \{C_j | 1 \leq j \leq n\}$. For a linear operator T on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$, define $T^*(A) = [T(A)]^*$ for all A in $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$. Let $T^*(R_i) = \{T^*(E_{ij}) | 1 \leq j \leq n\}$ for all $i = 1, \dots, m$ and $T^*(C_j) = \{T^*(E_{ij}) | 1 \leq i \leq m\}$ for all $j = 1, \dots, n$.

Lemma 2.4. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$. Suppose that T preserves rank 1 and perimeters 2 and $p (\geq 3)$ of rank-1 matrices. Then:*

- (1) *T maps two distinct cells in a row (or a column) into two distinct cells in a row or in a column with nonzero scalar multiplication;*
- (2) *For the case $m = n$, if T maps a row of a matrix A into a row with nonzero scalar multiplication, then T maps each row of A into a row of $T(A)$ with nonzero scalar multiplication. Similarly, if T maps a column of a matrix A into a column with nonzero scalar multiplication, then T maps each column of A into a column of $T(A)$ with nonzero scalar multiplication.*

Proof. (1): Suppose $T(E_{ij}) = \alpha E_{rl}$ and $T(E_{ih}) = \beta E_{rl}$ for some distinct pairs $(i, j) \neq (i, h)$ and some nonzero scalars $\alpha, \beta \in \mathbb{R}_{\max}$. Then T maps the i th row of a matrix A into r th row or l th column with scalar multiplication by Lemma 2.2. Without loss of generality, we assume the former. Thus for any rank-1 matrix A with perimeter $p (\geq 3)$ which dominates $E_{ij} \oplus E_{ih}$, we can show that $T(A)$ has perimeter at most $p - 1$, a contradiction. Thus T maps two distinct cells in a row into two distinct cells in a row or in a column with nonzero scalar multiplication.

(2): If not, then there exist rows R_i and R_j such that $T^*(R_i) \subseteq R_r$ and $T^*(R_j) \subseteq C_s$ for some r, s . Consider a rank-1 matrix $D = E_{ip} \oplus E_{iq} \oplus E_{jp} \oplus E_{jq}$ with $p \neq q$. Then we have

$$\begin{aligned} T(D) &= T(E_{ip} \oplus E_{iq}) \oplus T(E_{jp} \oplus E_{jq}) \\ &= (\alpha_1 E_{rp'} \oplus \alpha_2 E_{rq'}) \oplus (\beta_1 E_{p''s} \oplus \beta_2 E_{q''s}) \end{aligned}$$

for some $p' \neq q'$ and $p'' \neq q''$ and some nonzero scalars $\alpha_i, \beta_i \in \mathbb{R}_{\max}$ by (1). Therefore, $r(T(D)) \neq 1$ and T does not preserve rank 1, a contradiction. Hence T maps each row of A into a row (or a column) of $T(A)$ with nonzero scalar multiplication. Similarly, T maps each column of A into a column (or a row) of $T(A)$ with nonzero scalar multiplication. ■

Now we have an interesting example:

Example 2.5. Consider a linear operator T on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$, with $m \geq 3$ and $n \geq 4$, such that

$$T(A) = B = [b_{ij}],$$

where $A = [a_{i,j}]$ in $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$, $b_{i,j} = 0$ if $i \geq 2$ and $b_{i,j} = \bigoplus_{i=1}^m a_{i,r}$ with $r \equiv i + (j - 1) \pmod{n}$ and $1 \leq r \leq n$. Then T maps each row and each column into the first row with some scalar multiplication. And T preserves both rank and perimeters 2, 3 and $n + 1$ of rank-1 matrices. But T does not preserve perimeters k ($k \geq 4$ and $k \neq n + 1$) of rank-1 matrices:

For if $4 \leq k \leq n$, then we can choose a $2 \times (k - 2)$ submatrix with perimeter k which is mapped to distinct k position in the first row of B under T . Then this $1 \times k$ submatrix has perimeter $k + 1$. Therefore T does not preserve perimeter k of rank-1 matrices. ■

For a linear operator T on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ preserving rank 1 and perimeter 2 of rank-1 matrices, we define the corresponding mapping $T' : \Delta_{m,n} \rightarrow \Delta_{m,n}$ by $T'(i, j) = (k, l)$ whenever $T(E_{ij}) = b_{ij}E_{kl}$ for some nonzero scalar $b_{ij} \in \mathbb{R}_{\max}$. Then T' is well-defined by Lemma 2.2 (1).

Lemma 2.6. *Let T be a linear operator preserving both rank and perimeters 2 and $k(k \geq 4, k \neq n + 1)$ of rank-1 matrices. Then T' is a bijection on $\Delta_{m,n}$.*

Proof. By Lemma 2.2, $T(E_{ij}) = b_{ij}E_{rl}$ for some $(r, l) \in \Delta_{m,n}$ and some nonzero scalar $b_{ij} \in \mathbb{R}_{\max}$. Without loss of generality, we may assume that T maps the i th row of a matrix into the r th row with nonzero scalar multiplication. Suppose $T'(i, j) = T'(p, q)$ for some distinct pairs $(i, j), (p, q) \in \Delta_{m,n}$. By the definition of T' , we have $T(E_{ij}) = b_{ij}E_{rl}$ and $T(E_{pq}) = c_{pq}E_{rl}$ for some nonzero scalars $b_{ij}, c_{pq} \in \mathbb{R}_{\max}$. If $i = p$ or $j = q$, then we have contradictions by Lemma 2.4. So let $i \neq p$ and $j \neq q$.

If $k = n + k' \geq n + 2$, consider the matrix

$$D = \bigoplus_{s=1}^n E_{is} + \bigoplus_{t=1}^n E_{pt} + \bigoplus_{h=1}^{k'-2} \bigoplus_{g=1}^n E_{hg}$$

with rank 1 and perimeter $n + k' = k$. Then T maps the i th and p th row of D into the r th row with nonzero scalar multiplication by Lemma 2.4. Thus the perimeter of $T(D)$ is less than $n + k' = k$, a contradiction.

If $4 \leq k \leq n$, we will show that we can choose a $2 \times (k - 2)$ submatrix from the i th and p th row whose image under T has $1 \times k$ submatrix in the r th row as follows: Since $T(E_{ij}) = b_{ij}E_{rl}$ and $T(E_{pq}) = c_{pq}E_{rl}$, T maps the i th row and the p th row into the r th row. But T maps distinct cells in each row (or column) to distinct cells by Lemma 2.4. Now, choose E_{ij} ,

E_{pj} but do not choose E_{iq}, E_{pq} . Since there is a cell E_{ph} ($h \neq j, q$) in the p th row such that $T'(p, h) = T'(i, q)$ but $T'(i, h) \neq T'(p, j)$, we choose 2×2 submatrix $E_{ij} \oplus E_{ih} \oplus E_{pj} \oplus E_{ph}$ whose image under T is 1×4 submatrix in the r th row. And we can choose a cell E_{ps} ($s \neq q, j, h$) such that $T'(i, s) \neq T'(p, j), T'(p, q), T'(p, h)$. Then we have 2×3 submatrix $E_{ij} \oplus E_{ih} \oplus E_{is} \oplus E_{pj} \oplus E_{ph} \oplus E_{ps}$ whose image under T is 1×5 submatrix in the r th row. Similarly, we can choose a $2 \times (k-2)$ submatrix whose image under T is a $1 \times k$ submatrix in the r th row. This shows that T does not preserve the perimeter k of a rank-1 matrix, a contradiction.

Hence $T'(i, j) \neq T'(p, q)$ for any two distinct pairs $(i, j), (p, q) \in \Delta_{m,n}$. Therefore, T' is a bijection. \blacksquare

We obtain the following characterization theorem for linear operators preserving the rank and the perimeter of rank-1 matrices over max algebra.

Theorem 2.7. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$. Then the following are equivalent:*

- (1) T is a (U, V) -operator;
- (2) T preserves both rank and perimeter of rank-1 matrices;
- (3) T preserves both rank and perimeters 2 and k ($k \geq 4, k \neq n+1$) of rank-1 matrices.

Proof. (1) implies (2) by Proposition 2.1. It is obvious that (2) implies (3). We now show that (3) implies (1). Assume (3). Then the corresponding mapping $T' : \Delta_{m,n} \rightarrow \Delta_{m,n}$ is a bijection by Lemma 2.6.

By Lemma 2.4, there are two cases; (a) T^* maps \mathcal{R} onto \mathcal{R} and maps \mathcal{C} onto \mathcal{C} or (b) T^* maps \mathcal{R} onto \mathcal{C} and \mathcal{C} onto \mathcal{R} .

Case a): We note that $T^*(R_i) = R_{\sigma(i)}$ and $T^*(C_j) = C_{\tau(j)}$ for all i, j , where σ and τ are permutations of $\{1, \dots, m\}$ and $\{1, \dots, n\}$, respectively. Let P and Q be the permutation matrices corresponding to σ and τ , respectively. Then for any $E_{ij} \in \mathbb{E}_{m,n}$, we can write $T(E_{ij}) = b_{ij} E_{\sigma(i)\tau(j)}$ for some nonzero scalar $b_{ij} \in \mathbb{R}_{\max}$. Now we claim that for all $i, l \in \{1, \dots, m\}$ and all $j, r \in \{1, \dots, n\}$,

$$\frac{b_{ij}}{b_{ir}} = \frac{b_{lj}}{b_{lr}}.$$

Consider a matrix $E = E_{ij} \oplus E_{ik} \oplus E_{lj} \oplus E_{lr}$ with rank 1. Then we have

$$T(E) = b_{ij}E_{\sigma(i)\tau(j)} \oplus b_{ir}E_{\sigma(i)\tau(r)} \oplus b_{lj}E_{\sigma(l)\tau(j)} \oplus b_{lk}E_{\sigma(l)\tau(r)}.$$

Since $T(E)$ has rank 1, it follows that $\frac{b_{ij}}{b_{ir}} = \frac{b_{lj}}{b_{lk}}$. Let $C \in \mathcal{M}_{m,m}(\mathbb{R}_{\max})$ and $D \in \mathcal{M}_{n,n}(\mathbb{R}_{\max})$ be diagonal matrices such that $c_{11} = 1$, $d_{11} = b_{11}$, $c_{ii} = \frac{b_{i1}}{b_{11}}$, and $d_{jj} = b_{1j}$ for all $i = 2, \dots, m$ and $j = 2, \dots, n$. Then $b_{ij} = c_{ii}d_{jj}$ for all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

Let $A = [a_{ij}]$ be any $m \times n$ matrix in $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$. Then we have

$$\begin{aligned} T(A) &= T\left(\sum_{i=1}^m \sum_{j=1}^n a_{ij}E_{ij}\right) = \sum_{i=1}^m \sum_{j=1}^n a_{ij}T(E_{ij}) \\ &= \sum_{i=1}^m \sum_{j=1}^n a_{ij}b_{ij}E_{\sigma(i)\tau(j)} = \sum_{i=1}^m \sum_{j=1}^n c_{ii}a_{ij}E_{\sigma(i)\tau(j)}d_{jj} \\ &= C \otimes P \otimes A \otimes Q \otimes D. \end{aligned}$$

Since $C \otimes P = U$ is an $m \times m$ monomial and $Q \otimes D = V$ is an $n \times n$ monomial, it follows that T is a (U, V) -operator.

Case b): We note that $m = n$ and $T^*(R_i) = C_{\sigma(i)}$ and $T^*(C_j) = R_{\tau(j)}$ for all i, j , where σ and τ are permutations of $\{1, \dots, m\}$. By the similar argument of case a), we obtain that $T(A)$ is of the form $T(A) = C \otimes P \otimes A^t \otimes Q \otimes D$. Thus T is a (U, V) -operator. ■

Even though T is an invertible operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$, T may not preserve rank 1. For example, let $T : \mathcal{M}_{2,2}(\mathbb{R}_{\max}) \rightarrow \mathcal{M}_{2,2}(\mathbb{R}_{\max})$ by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ d & c \end{bmatrix}$. Then T is clearly invertible, but does not preserve rank 1 since $T\left(\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. However all (U, V) -operator is invertible and its inverse is (U^{-1}, V^{-1}) -operator. Here, U^{-1} and V^{-1} are monomials since U and V are monomials.

Corollary 2.8. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ that preserves both rank and perimeter of rank-1 matrices. Then T is invertible. ■*

We say that a linear operator T on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ *strongly preserves* perimeter k of rank-1 matrices if $P(T(A)) = k$ if and only if $P(A) = k$.

Consider a linear operator T on $\mathcal{M}_{2,2}(\mathbb{R}_{\max})$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a \oplus b \oplus c \oplus d) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then T preserves both rank and perimeter 2 of rank-1 matrices but does not strongly preserve perimeter 2, since $T\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\right) = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$ with $P\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\right) = 4$ but $P\left(\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}\right) = 2$.

Theorem 2.9. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$. Then T preserves both rank and perimeter of rank-1 matrices if and only if it preserves perimeter 3 and strongly preserves perimeter 2 of rank-1 matrices.*

Proof. Suppose T preserves perimeter 3 and strongly preserves perimeter 2 of rank-1 matrices. Then T maps each row of a matrix into a row or a column (if $m = n$) with nonzero scalar multiplication. Since T strongly preserves perimeter 2, T maps each cell onto a cell with nonzero scalar multiplication. This means that the corresponding mapping T' is a bijection. Thus, by the similar method as in the proof of Theorem 2.7, T preserves both rank and perimeter of rank-1 matrices.

The converse is immediate. ■

Theorem 2.10. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_{\max})$ that preserves the rank of rank-1 matrices. Then T preserves perimeter of rank-1 matrices if and only if it strongly preserves perimeter 2 of rank-1 matrices.*

Proof. Suppose T strongly preserves perimeter 2 of rank-1 matrices. Then T maps each cell onto a cell with nonzero scalar multiplication. Thus T' is a bijection. Since T preserves rank 1, it maps a row of a matrix into a row or a column (if $m = n$). Thus, by the similar method as in the proof of Theorem 2.7, T preserves both rank and perimeter of rank-1 matrices.

The converse is immediate. ■

Thus we have characterizations of the linear operators that preserve both rank and perimeter of rank-1 matrices over max algebra.

REFERENCES

- [1] R.B. Bapat, *A max version of the Perron-Frebenius theorem*, Linear Algebra Appl. **275–276** (1998), 3–18.
- [2] R.B. Bapat, S. Pati and S.-Z. Song, *Rank preservers of matrices over max algebra*, Linear and Multilinear Algebra **48** (2000), 149–164.
- [3] L.B. Beasley and N.J. Pullman, *Boolean rank-preserving operators and Boolean rank-1 spaces*, Linear Algebra Appl. **59** (1984), 55–77.
- [4] L.B. Beasley, S.-Z. Song and S.-G. Lee, *Zero term rank preservers*, Linear and Multilinear Algebra **48** (2001), 313–318.
- [5] S.-Z. Song and S.-R. Park, *Maximal column rank preservers of fuzzy matrices*, Discuss. Math. – Gen. Algebra Appl. **21** (2001), 207–218.

Received 24 April 2003