

REPRESENTABLE DUALY RESIDUATED LATTICE-ORDERED MONOIDS

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Abstract

Dually residuated lattice-ordered monoids (*DRℓ*-monoids) generalize lattice-ordered groups and include also some algebras related to fuzzy logic (e.g. *GMV*-algebras and pseudo *BL*-algebras). In the paper, we give some necessary and sufficient conditions for a *DRℓ*-monoid to be representable (i.e. a subdirect product of totally ordered *DRℓ*-monoids) and we prove that the class of representable *DRℓ*-monoids is a variety.

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Commutative dually residuated ℓ -monoids (*DRℓ*-semigroups) were introduced in [18] as a common generalization of Abelian lattice-ordered groups (ℓ -groups) and Brouwerian algebras. Likewise, well-known *MV*-algebras that constitute an algebraic counterpart of the Łukasiewicz logic and *BL*-algebras as algebras of Hájek's basic logic are contained among bounded *DRℓ*-semigroups (see [15] and [16]). Moreover, any *BL*-algebra (and hence any *MV*-algebra) is a representable *DRℓ*-semigroup, that is, a subdirect product of totally ordered *DRℓ*-semigroups. By [19], commutative representable *DRℓ*-semigroups are characterized by the identity $(x - y) \wedge (y - x) \leq 0$.

Non-commutative *DRℓ*-monoids embrace lattice-ordered groups as well as some algebras that are in close connection to fuzzy logic. For instance, pseudo *BL*-algebras and, in particular, *GMV*-algebras (called also pseudo

MV -algebras), i.e. non-commutative extensions of BL -algebras and MV -algebras, respectively, can be viewed as a particular kind of bounded $DR\ell$ -monoids (see [13] and [17]).

The objective of the present paper is the description of representable $DR\ell$ -monoids; it is shown that those form a variety.

Recall the notion of a (non-commutative) $DR\ell$ -monoid. An algebra $(A; +, 0, \vee, \wedge, \rightarrow, \leftarrow)$ of type $\langle 2, 0, 2, 2, 2, 2 \rangle$ is said to be a *dually residuated lattice-ordered monoid*, or simply a *$DR\ell$ -monoid*, if

- (i) $(A; +, 0, \vee, \wedge)$ is an ℓ -monoid, i.e., $(A; +, 0)$ is a monoid, $(A; \vee, \wedge)$ is a lattice and "+" distributes over " \vee " and " \wedge ";
- (ii) for any $x, y \in A$, $x \rightarrow y$ is the least $s \in A$ such that $s + y \geq x$, and $x \leftarrow y$ is the least $t \in A$ such that $y + t \geq x$; and
- (iii) A fulfils the identities

$$((x \rightarrow y) \vee 0) + y \leq x \vee y, \quad y + ((x \leftarrow y) \vee 0) \leq x \vee y.$$

In the original definition the validity of the inequalities $x \rightarrow x \geq 0$ and $x \leftarrow x \geq 0$ was also desired, but analogously as in [11], one can prove that we always have $x \rightarrow x = x \leftarrow x = 0$. Moreover, (iii) holds even with " \leq " substituted by " $=$ ". Notice next that the condition (ii) is equivalent to the following system of identities (see [17]):

$$\begin{aligned} (x \rightarrow y) + y &\geq x, \quad y + (x \leftarrow y) \geq x, \\ x \rightarrow y &\leq (x \vee z) \rightarrow y, \quad x \leftarrow y \leq (x \vee z) \leftarrow y, \\ (x + y) \rightarrow y &\leq x, \quad (y + x) \leftarrow y \leq x. \end{aligned}$$

Thus $DR\ell$ -monoids form an equational class. Some properties of this variety were examined in [10].

Let us mention several concepts and facts from [12], [13] and [14]. For basic properties of non-commutative $DR\ell$ -monoids see [10] or [12].

For any x of a $DR\ell$ -monoid A , the *absolute value* of x is defined by $|x| = x \vee (0 \rightarrow x)$, or equivalently $|x| = x \vee (0 \leftarrow x)$, and $x^+ = x \vee 0$ is the *positive part* of x . For each $X \subseteq A$, let $X^+ = \{x \in X : x \geq 0\}$.

A subset H of A is said to be an *ideal* of A if it satisfies the following conditions:

- (I1) $0 \in H$;
- (I2) if $x, y \in H$, then $x + y \in H$;
- (I3) for all $x \in H$ and $y \in A$, $|y| \leq |x|$ implies $y \in H$.

Under the ordering by set inclusion, the set of all ideals of any $DR\ell$ -monoid becomes an algebraic, distributive lattice $\mathcal{I}(A)$.

An ideal H of A is said to be *prime* if it is a finitely meet-irreducible element of the lattice $\mathcal{I}(A)$ of all ideals of A , i.e., if $H = J \cap K$, then $H = J$ or $H = K$ for all $J, K \in \mathcal{I}(A)$. The prime ideals play an important role in the study of ideals since each ideal is an intersection of prime ideals. The assumption of the validity of the identities

$$(*) \quad \begin{aligned} (x \multimap y)^+ \wedge (y \multimap x)^+ &= 0 \\ (x \multimapleft y)^+ \wedge (y \multimapleft x)^+ &= 0 \end{aligned}$$

makes it possible to prove the following useful characterization of prime ideals. Let us note that the conditions (i) through (iv) are equivalent in any $DR\ell$ -monoid.

Lemma 1 [14]. *If A satisfies $(*)$, then for any ideal H , the following statements are equivalent (for all $J, K \in \mathcal{I}(A)$ and $x, y \in A$):*

- (i) H is prime;
- (ii) if $J \cap K \subseteq H$, then $J \subseteq H$ or $K \subseteq H$;
- (iii) if $|x| \wedge |y| \in H$, then $x \in H$ or $y \in H$;
- (iv) if $x \wedge y \in H^+$, then $x \in H$ or $y \in H$;
- (v) if $x \wedge y \in H$, then $x \in H$ or $y \in H$;
- (vi) if $x \wedge y = 0$, then $x \in H$ or $y \in H$;
- (vii) $(x \multimap y)^+ \in H$ or $(y \multimap x)^+ \in H$;
- (viii) $(x \multimapleft y)^+ \in H$ or $(y \multimapleft x)^+ \in H$;
- (ix) the set of all ideals exceeding H is totally ordered by inclusion. ■

Since $(*)$ holds in any ℓ -group, in any linearly ordered (and hence in any representable) $DR\ell$ -monoid and in any bounded $DR\ell$ -monoid, which is induced by a GMV -algebra or by a pseudo BL -algebra, respectively (see [17] and [13]), the previous lemma describes the prime ideals (respectively, prime filters in the case of pseudo BL -algebras) in the mentioned algebras.

We say that an ideal H of a $DR\ell$ -monoid A is *normal* if $x + H^+ = H^+ + x$ for all $x \in A$. For any $H \in \mathcal{I}(A)$, H is normal if and only if $(x \rightarrow y)^+ \in H$ iff $(x \leftarrow y)^+ \in H$ for all $x, y \in A$.

As it was proved in [12], the normal ideals of any $DR\ell$ -monoid correspond one-to-one to its congruence relations; the lattice $\mathcal{N}(A)$ of all normal ideals is isomorphic with $\text{Con}(A)$, the congruence lattice of A .

The next lemma states an important property of normal prime ideals:

Lemma 2. *Let A be a $DR\ell$ -monoid satisfying $(*)$ and H be a normal ideal. Then A/H is totally ordered if and only if H is prime. ■*

Let A be a $DR\ell$ -monoid and $X \subseteq A$. The set

$$X^\perp = \{a \in A : |a| \wedge |x| = 0 \text{ for all } x \in X\}$$

is called the *polar* of X . For any $a \in A$, we write briefly a^\perp instead of $\{a\}^\perp$. A subset X of A is a *polar in A* if $X = Y^\perp$ for some $Y \subseteq A$.

By [14], X^\perp is equal to the intersection of all minimal prime ideals not containing X and hence any polar is an ideal in A . In addition, the polars are just the pseudocomplements in the ideal lattice $\mathcal{I}(A)$.

Let $\{A_i\}_{i \in I}$ be a collection of $DR\ell$ -monoids. Recall that A is a *subdirect product* of $\{A_i\}_{i \in I}$ if there is an embedding φ of A into the direct product $\prod_{i \in I} A_i$ such that the homomorphisms $\varphi\pi_i$ map A onto A_i for all $i \in I$, where π_i is the natural projection of $\prod_{i \in I} A_i$ onto A_i .

A $DR\ell$ -monoid is said to be *representable* if it is a subdirect product of linearly ordered $DR\ell$ -monoids. Note that representable ℓ -groups are also called *residually ordered ℓ -groups* (see, e.g., [7]).

If a $DR\ell$ -monoid fulfils $(*)$, then, by Lemma 2, its subdirect representations by totally ordered $DR\ell$ -monoids are associated with families of normal prime ideals whose intersections are precisely $\{0\}$. Therefore, it is obvious that every commutative $DR\ell$ -monoid satisfying $(x - y) \wedge (y - x) \leq 0$ is representable. On the contrary, this fails in the case of non-commutative $DR\ell$ -monoids. For instance, any ℓ -group G is a $DR\ell$ -monoid with $(*)$, however, G need not be representable (residually ordered).

Lemma 3. *If P is a minimal prime ideal of a DR ℓ -monoid A , then $A^+ \setminus P$ is a maximal filter of the lattice $(A^+; \vee, \wedge)$.*

Proof. By Zorn's Lemma, there exists a maximal filter F of $(A^+; \vee, \wedge)$ with $A^+ \setminus P \subseteq F$. (Since P^+ is also a prime ideal of $(A^+; \vee, \wedge)$, it follows that $A^+ \setminus P = A^+ \setminus P^+$ is a prime filter of $(A^+; \vee, \wedge)$ which is contained in some maximal filter.) The aim of the proof is to show $F = A^+ \setminus P^+$.

We claim that $P^+ = Q^+$, where $Q = \bigcup\{a^\perp : a \in F\}$.

If $x \in Q^+$, that is, $x \wedge a = 0$ for some $a \in F$, then $x \notin F$. Indeed, if $x \in F$, then $0 = x \wedge a \in F$ which entails $F = A^+$. Thus $x \in A^+ \setminus F \subseteq A^+ \setminus (A^+ \setminus P^+) = P^+$, whence $Q^+ \subseteq A^+ \setminus F \subseteq P^+$.

We shall now prove that Q is a prime ideal of A .

(I1): Since any principal polar a^\perp contains 0, so deos Q .

(I2): If $x, y \in Q$, i.e., $|x| \wedge a = 0$ and $|y| \wedge b = 0$ for some $a, b \in F$, then $0 \leq |x+y| \wedge a \wedge b \leq (|x|+|y|+|x|) \wedge a \wedge b \leq (|x| \wedge a \wedge b) + (|y| \wedge a \wedge b) + (|x| \wedge a \wedge b) = 0$. Therefore $x + y \in (a \wedge b)^\perp \subseteq Q$.

(I3): It is obvious since a^\perp is an ideal of A for each $a \in A$.

In order to prove that Q is prime, suppose that $x \wedge y \in Q^+$ yet $x \notin Q$, that is, $x \wedge y \wedge a = 0$ for some $a \in F$, and $x \wedge a > 0$ for all $a \in F$. If $x \notin F$, then the filter of $(A^+; \vee, \wedge)$ generated by $F \cup \{x\}$ is equal to A^+ , and hence $0 \geq a \wedge x$ for some $a \in F$, a contradiction. Therefore $x \in F$, and so $x \wedge a \in F$ which yields $y \in (x \wedge a)^\perp \subseteq Q$.

Let $x \in Q$; then $|x| \in Q^+ \subseteq P^+$, whence $x \in P$ showing $Q \subseteq P$. However, P is minimal prime; so $Q = P$. Hence $P^+ = Q^+$ as claimed. This gives $P^+ = A^+ \setminus F$, and consequently, $F = A^+ \setminus P^+$. ■

Observe that we have shown somewhat more than stated:

Lemma 4. *A prime ideal P of a DR ℓ -monoid A is minimal if and only if*

$$P = \bigcup\{a^\perp : a \in A^+ \setminus P\}.$$

Proof. By the proof of the previous lemma, $P = \bigcup\{a^\perp : a \in F\}$, where $F = A^+ \setminus P^+$.

Conversely, suppose that $Q \subseteq P$ for some prime ideal Q . If $Q \neq P$, then there is $x \in P^+ \setminus Q$, i.e., $x \in a^\perp$ for some $a \in A^+ \setminus P$. Since $x \wedge a = 0$, $x \notin Q$ and Q is prime, it follows that $a \in Q \subseteq P$, a contradiction. Thus $Q = P$. ■

The following results generalize the analogous properties of ℓ -groups, pseudo MV-algebras (GMV-algebras) and pseudo BL-algebras (see [7], [6], [4], [5] and [13]):

Theorem 5. *For any DRl-monoid A satisfying $(*)$, the following statements are equivalent:*

- (i) A is representable.
- (ii) There exists a family $\{P_i\}_{i \in I}$ of normal prime ideals of A such that

$$\bigcap_{i \in I} P_i = \{0\}.$$

- (iii) Every polar is a normal ideal.
- (iv) For any $a \in A^+$, $a^\perp \in \mathcal{N}(A)$.
- (v) Every minimal prime ideal is normal.

Proof. As argued above, the equivalence of (i) and (ii) is clear.

(i) \Rightarrow (iii). Suppose that A is a subdirect product of linearly ordered DRl-monoids $\{A_i\}_{i \in I}$. Observe that

$$(1) \quad x \wedge y = 0 \text{ iff } \{i \in I : x_i > 0_i\} \cap \{i \in I : y_i > 0_i\} = \emptyset$$

for all $x, y \in A^+$, as all A_i are linearly ordered.

Let now P be a polar in A , i.e., $P = P^{\perp\perp}$. Let $x \in A$, $a \in P^+$ and $y \in P^\perp$. Then $x + a \geq x$ implies $x + a = (x + a) \vee x = ((x + a) \rightarrow x) + x$. Further,

$$\{i \in I : (x_i + a_i) \rightarrow x_i > 0_i\} \subseteq \{i \in I : a_i > 0_i\}.$$

Indeed, if $a_i = 0_i$, then $(x_i + a_i) \rightarrow x_i = x_i \rightarrow x_i = 0_i$. Hence, we obtain

$$\begin{aligned} \{i \in I : (x_i + a_i) \rightarrow x_i > 0_i\} \cap \{i \in I : |y_i| > 0_i\} &\subseteq \\ &\subseteq \{i \in I : a_i > 0_i\} \cap \{i \in I : |y_i| > 0_i\} = \emptyset, \end{aligned}$$

by (1), since $a \wedge |y| = 0$. Therefore, $((x + a) \rightarrow x) \wedge |y| = 0$, and thus $(x + a) \rightarrow x \in P^{\perp\perp} = P$. Hence, $x + a = ((x + a) \rightarrow x) + x \in P^+ + x$ proving $x + P^+ \subseteq P^+ + x$. One analogously proves the other inclusion.

The implication (iii) \Rightarrow (iv) is obvious and (iv) implies (v) immediately by Lemma 4.

(v) \Rightarrow (ii). Since every prime ideal contains a minimal prime ideal and the intersection of all prime ideals equals to $\{0\}$, so does the intersection of all minimal prime ideals. Thus A is representable. \blacksquare

Theorem 6. *A DRℓ-monoid is representable if and only if it satisfies the identities*

$$(2) \quad (x \rightarrow y)^+ \wedge (((y \rightarrow x)^+ + z) \leftarrow z) = 0,$$

$$(3) \quad (x \leftarrow y)^+ \wedge ((z + (y \leftarrow x)^+) \rightarrow z) = 0.$$

Proof. Any linearly ordered DRℓ-monoid satisfies (2) and (3) since either $(x \rightarrow y)^+ = 0$ or $(y \rightarrow x)^+ = 0$ (respectively, either $(x \leftarrow y)^+ = 0$ or $(y \leftarrow x)^+ = 0$). Therefore the part "only if" is obvious.

Conversely, suppose that the above identities are satisfied by A ; then A fulfils also (*) (put $z = 0$). In view of Theorem 5, it suffices to prove that x^\perp is normal for all $x \in A^+$.

Let $y \in (x^\perp)^+$, that is, $y \wedge x = 0$. Observe that in this case

$$x = x \rightarrow (y \wedge x) = (x \rightarrow y) \vee (x \rightarrow x) = (x \rightarrow y) \vee 0 = (x \rightarrow y)^+,$$

and similarly $y = (y \rightarrow x)^+$. Hence, by (2),

$$x \wedge ((y + z) \leftarrow z) = (x \rightarrow y)^+ \wedge (((y \rightarrow x)^+ + z) \leftarrow z) = 0;$$

thus $(y + z) \leftarrow z \in (x^\perp)^+$. Further, $y + z \geq z$ implies $y + z = (y + z) \vee z = z + ((y + z) \leftarrow z) \in z + (x^\perp)^+$ which shows $(x^\perp)^+ + z \subseteq z + (x^\perp)^+$. The other inclusion follows similarly by (3). ■

Corollary 7. *A DRℓ-monoid is representable if and only if it satisfies the identities*

$$(x \rightarrow y) \wedge (((y \rightarrow x) + z) \leftarrow z) \leq 0,$$

$$(x \leftarrow y) \wedge ((z + (y \leftarrow x)) \rightarrow z) \leq 0.$$

Proof. One readily sees that

$$(x \rightarrow y)^+ \wedge (((y \rightarrow x)^+ + z) \leftarrow z) = [(x \rightarrow y) \wedge (((y \rightarrow x) + z) \leftarrow z)]^+,$$

$$(x \leftarrow y)^+ \wedge ((z + (y \leftarrow x)^+) \rightarrow z) = [(x \leftarrow y) \wedge ((z + (y \leftarrow x)) \rightarrow z)]^+.$$

■

Corollary 8. *The class of all representable DRℓ-monoids is a proper subvariety of the variety of all DRℓ-monoids.*

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