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REPRESENTABLE DUALLY RESIDUATED LATTICE-ORDERED MONOIDS

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Abstract

Dually residuated lattice-ordered monoids ($DR\ell$-monoids) generalize lattice-ordered groups and include also some algebras related to fuzzy logic (e.g. GMV-algebras and pseudo BL-algebras). In the paper, we give some necessary and sufficient conditions for a $DR\ell$-monoid to be representable (i.e. a subdirect product of totally ordered $DR\ell$-monoids) and we prove that the class of representable $DR\ell$-monoids is a variety.

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Commutative dually residuated ℓ-monoids ($DR\ell$-semigroups) were introduced in [18] as a common generalization of Abelian lattice-ordered groups (ℓ-groups) and Brouwerian algebras. Likewise, well-known MV-algebras that constitute an algebraic counterpart of the Łukasiewicz logic and BL-algebras as algebras of Hájek's basic logic are contained among bounded $DR\ell$-semigroups (see [15] and [16]). Moreover, any BL-algebra (and hence any MV-algebra) is a representable $DR\ell$-semigroup, that is, a subdirect product of totally ordered $DR\ell$-semigroups. By [19], commutative representable $DR\ell$-semigroups are characterized by the identity $(x - y) \wedge (y - x) \leq 0$.

Non-commutative $DR\ell$-monoids embrace lattice-ordered groups as well as some algebras that are in close connection to fuzzy logic. For instance, pseudo BL-algebras and, in particular, GMV-algebras (called also pseudo

MV-algebras), i.e. non-commutative extensions of BL-algebras and MV-algebras, respectively, can be viewed as a particular kind of bounded $DR\ell$-monoids (see [13] and [17]).

The objective of the present paper is the description of representable $DR\ell$-monoids; it is shown that those form a variety.

Recall the notion of a (non-commutative) $DR\ell$-monoid. An algebra $(A; +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ of type $\langle 2, 0, 2, 2, 2, 2\rangle$ is said to be a *dually residuated lattice-ordered monoid*, or simply a *$DR\ell$-monoid*, if

(i) $(A; +, 0, \vee, \wedge)$ is an ℓ-monoid, i.e., $(A; +, 0)$ is a monoid, $(A; \vee, \wedge)$ is a lattice and "$+$" distributes over "$\vee$" and "$\wedge$";

(ii) for any $x, y \in A$, $x \rightharpoonup y$ is the least $s \in A$ such that $s + y \geq x$, and $x \leftharpoondown y$ is the least $t \in A$ such that $y + t \geq x$; and

(iii) A fulfils the identities

$$((x \rightharpoonup y) \vee 0) + y \leq x \vee y, \;\; y + ((x \leftharpoondown y) \vee 0) \leq x \vee y.$$

In the original definition the validity of the inequalities $x \rightharpoonup x \geq 0$ and $x \leftharpoondown x \geq 0$ was also desired, but analogously as in [11], one can prove that we always have $x \rightharpoonup x = x \leftharpoondown x = 0$. Moreover, (iii) holds even with "$\leq$" substituted by "$=$". Notice next that the condition (ii) is equivalent to the following system of identities (see [17]):

$$(x \rightharpoonup y) + y \geq x, \;\; y + (x \leftharpoondown y) \geq x,$$

$$x \rightharpoonup y \leq (x \vee z) \rightharpoonup y, \;\; x \leftharpoondown y \leq (x \vee z) \leftharpoondown y,$$

$$(x + y) \rightharpoonup y \leq x, \;\; (y + x) \leftharpoondown y \leq x.$$

Thus $DR\ell$-monoids form an equational class. Some properties of this variety were examined in [10].

Let us mention several concepts and facts from [12], [13] and [14]. For basic properties of non-commutative $DR\ell$-monoids see [10] or [12].

For any x of a $DR\ell$-monoid A, the *absolute value* of x is defined by $|x| = x \vee (0 \rightharpoonup x)$, or equivalently $|x| = x \vee (0 \leftharpoondown x)$, and $x^+ = x \vee 0$ is the *positive part* of x. For each $X \subseteq A$, let $X^+ = \{x \in X : x \geq 0\}$.

A subset H of A is said to be an *ideal* of A if it satisfies the following conditions:

(I1) $0 \in H$;

(I2) if $x, y \in H$, then $x + y \in H$;

(I3) for all $x \in H$ and $y \in A$, $|y| \leq |x|$ implies $y \in H$.

Under the ordering by set inclusion, the set of all ideals of any $DR\ell$-monoid becomes an algebraic, distributive lattice $\mathcal{I}(A)$.

An ideal H of A is said to be *prime* if it is a finitely meet-irreducible element of the lattice $\mathcal{I}(A)$ of all ideals of A, i.e., if $H = J \cap K$, then $H = J$ or $H = K$ for all $J, K \in \mathcal{I}(A)$. The prime ideals play an important role in the study of ideals since each ideal is an intersection of prime ideals. The assumption of the validity of the identities

$$(*) \qquad \begin{aligned} (x \rightharpoonup y)^+ \wedge (y \rightharpoonup x)^+ &= 0 \\ (x \leftharpoondown y)^+ \wedge (y \leftharpoondown x)^+ &= 0 \end{aligned}$$

makes it possible to prove the following useful characterization of prime ideals. Let us note that the conditions (i) through (iv) are equivalent in any $DR\ell$-monoid.

Lemma 1 [14]. *If A satisfies $(*)$, then for any ideal H, the following statements are equivalent (for all $J, K \in \mathcal{I}(A)$ and $x, y \in A$):*

(i) *H is prime;*

(ii) *if $J \cap K \subseteq H$, then $J \subseteq H$ or $K \subseteq H$;*

(iii) *if $|x| \wedge |y| \in H$, then $x \in H$ or $y \in H$;*

(iv) *if $x \wedge y \in H^+$, then $x \in H$ or $y \in H$;*

(v) *if $x \wedge y \in H$, then $x \in H$ or $y \in H$;*

(vi) *if $x \wedge y = 0$, then $x \in H$ or $y \in H$;*

(vii) *$(x \rightharpoonup y)^+ \in H$ or $(y \rightharpoonup x)^+ \in H$;*

(viii) *$(x \leftharpoondown y)^+ \in H$ or $(y \leftharpoondown x)^+ \in H$;*

(ix) *the set of all ideals exceeding H is totally ordered by inclusion.* ∎

Since $(*)$ holds in any ℓ-group, in any linearly ordered (and hence in any representable) $DR\ell$-monoid and in any bounded $DR\ell$-monoid, which is induced by a GMV-algebra or by a pseudo BL-algebra, respectively (see [17] and [13]), the previous lemma describes the prime ideals (respectively, prime filters in the case of pseudo BL-algebras) in the mentioned algebras.

We say that an ideal H of a $DR\ell$-monoid A is *normal* if $x+H^+ = H^+ +x$ for all $x \in A$. For any $H \in \mathcal{I}(A)$, H is normal if and only if $(x \rightharpoonup y)^+ \in H$ iff $(x \leftharpoondown y)^+ \in H$ for all $x, y \in A$.

As it was proved in [12], the normal ideals of any $DR\ell$-monoid correspond one-to-one to its congruence relations; the lattice $\mathcal{N}(A)$ of all normal ideals is isomorphic with $\mathrm{Con}(A)$, the congruence lattice of A.

The next lemma states an important property of normal prime ideals:

Lemma 2. *Let A be a $DR\ell$-monoid satisfying $(*)$ and H be a normal ideal. Then A/H is totally ordered if and only if H is prime.* ■

Let A be a $DR\ell$-monoid and $X \subseteq A$. The set

$$X^\perp = \{a \in A : |a| \wedge |x| = 0 \text{ for all } x \in X\}$$

is called the *polar of* X. For any $a \in A$, we write briefly $a^\perp$ instead of $\{a\}^\perp$. A subset X of A is a *polar in* A if $X = Y^\perp$ for some $Y \subseteq A$.

By [14], $X^\perp$ is equal to the intersection of all minimal prime ideals not containing X and hence any polar is an ideal in A. In addition, the polars are just the pseudocomplements in the ideal lattice $\mathcal{I}(A)$.

Let $\{A_i\}_{i\in I}$ be a collection of $DR\ell$-monoids. Recall that A is a *subdirect product* of $\{A_i\}_{i\in I}$ if there is an embedding φ of A into the direct product $\prod_{i\in I} A_i$ such that the homomorphisms $\varphi\pi_i$ map A onto A_i for all $i \in I$, where π_i is the natural projection of $\prod_{i\in I} A_i$ onto A_i.

A $DR\ell$-monoid is said to be *representable* if it is a subdirect product of linearly ordered $DR\ell$-monoids. Note that representable ℓ-groups are also called *residually ordered* ℓ-groups (see, e.g., [7]).

If a $DR\ell$-monoid fulfils $(*)$, then, by Lemma 2, its subdirect representations by totally ordered $DR\ell$-monoids are associated with families of normal prime ideals whose intersections are precisely $\{0\}$. Therefore, it is obvious that every commutative $DR\ell$-monoid satisfying $(x-y)\wedge(y-x) \leq 0$ is representable. On the contrary, this fails in the case of non-commutative $DR\ell$-monoids. For instance, any ℓ-group G is a $DR\ell$-monoid with $(*)$, however, G need not be representable (residually ordered).

Lemma 3. *If P is a minimal prime ideal of a DRℓ-monoid A, then $A^+ \setminus P$ is a maximal filter of the lattice $(A^+; \vee, \wedge)$.*

Proof. By Zorn's Lemma, there exists a maximal filter F of $(A^+; \vee, \wedge)$ with $A^+ \setminus P \subseteq F$. (Since P^+ is also a prime ideal of $(A^+; \vee, \wedge)$, it follows that $A^+ \setminus P = A^+ \setminus P^+$ is a prime filter of $(A^+; \vee, \wedge)$ which is contained in some maximal filter.) The aim of the proof is to show $F = A^+ \setminus P^+$.

We claim that $P^+ = Q^+$, where $Q = \bigcup\{a^\perp : a \in F\}$.

If $x \in Q^+$, that is, $x \wedge a = 0$ for some $a \in F$, then $x \notin F$. Indeed, if $x \in F$, then $0 = x \wedge a \in F$ which entails $F = A^+$. Thus $x \in A^+ \setminus F \subseteq A^+ \setminus (A^+ \setminus P^+) = P^+$, whence $Q^+ \subseteq A^+ \setminus F \subseteq P^+$.

We shall now prove that Q is a prime ideal of A.

(I1): Since any principal polar $a^\perp$ contains 0, so deos Q.

(I2): If $x, y \in Q$, i.e., $|x| \wedge a = 0$ and $|y| \wedge b = 0$ for some $a, b \in F$, then $0 \leq |x+y| \wedge a \wedge b \leq (|x|+|y|+|x|) \wedge a \wedge b \leq (|x| \wedge a \wedge b)+(|y| \wedge a \wedge b)+(|x| \wedge a \wedge b) = 0$. Therefore $x + y \in (a \wedge b)^\perp \subseteq Q$.

(I3): It is obvious since $a^\perp$ is an ideal of A for each $a \in A$.

In order to prove that Q is prime, suppose that $x \wedge y \in Q^+$ yet $x \notin Q$, that is, $x \wedge y \wedge a = 0$ for some $a \in F$, and $x \wedge a > 0$ for all $a \in F$. If $x \notin F$, then the filter of $(A^+; \vee, \wedge)$ generated by $F \cup \{x\}$ is equal to A^+, and hence $0 \geq a \wedge x$ for some $a \in F$, a contradiction. Therefore $x \in F$, and so $x \wedge a \in F$ which yields $y \in (x \wedge a)^\perp \subseteq Q$.

Let $x \in Q$; then $|x| \in Q^+ \subseteq P^+$, whence $x \in P$ showing $Q \subseteq P$. However, P is minimal prime; so $Q = P$. Hence $P^+ = Q^+$ as claimed. This gives $P^+ = A^+ \setminus F$, and consequently, $F = A^+ \setminus P^+$. ■

Observe that we have shown somewhat more than stated:

Lemma 4. *A prime ideal P of a DRℓ-monoid A is minimal if and only if*

$$P = \bigcup\{a^\perp : a \in A^+ \setminus P\}.$$

Proof. By the proof of the previous lemma, $P = \bigcup\{a^\perp : a \in F\}$, where $F = A^+ \setminus P^+$.

Conversely, suppose that $Q \subseteq P$ for some prime ideal Q. If $Q \neq P$, then there is $x \in P^+ \setminus Q$, i.e., $x \in a^\perp$ for some $a \in A^+ \setminus P$. Since $x \wedge a = 0$, $x \notin Q$ and Q is prime, it follows that $a \in Q \subseteq P$, a contradiction. Thus $Q = P$. ■

The following results generalize the analogous properties of ℓ-groups, pseudo MV-algebras (GMV-algebras) and pseudo BL-algebras (see [7], [6], [4], [5] and [13]):

Theorem 5. *For any $DR\ell$-monoid A satisfying $(*)$, the following statements are equivalent:*

(i) *A is representable.*

(ii) *There exists a family $\{P_i\}_{i\in I}$ of normal prime ideals of A such that*

$$\bigcap_{i\in I} P_i = \{0\}.$$

(iii) *Every polar is a normal ideal.*

(iv) *For any $a \in A^+$, $a^\perp \in \mathcal{N}(A)$.*

(v) *Every minimal prime ideal is normal.*

Proof. As argued above, the equivalence of (i) and (ii) is clear.

(i) $\Rightarrow$ (iii). Suppose that A is a subdirect product of linearly ordered $DR\ell$-monoids $\{A_i\}_{i\in I}$. Observe that

$$x \wedge y = 0 \text{ iff } \{i \in I : x_i > 0_i\} \cap \{i \in I : y_i > 0_i\} = \varnothing \tag{1}$$

for all $x, y \in A^+$, as all A_i are linearly ordered.

Let now P be a polar in A, i.e., $P = P^{\perp\perp}$. Let $x \in A$, $a \in P^+$ and $y \in P^\perp$. Then $x + a \geq x$ implies $x + a = (x + a) \vee x = ((x + a) \rightharpoonup x) + x$. Further,

$$\{i \in I : (x_i + a_i) \rightharpoonup x_i > 0_i\} \subseteq \{i \in I : a_i > 0_i\}.$$

Indeed, if $a_i = 0_i$, then $(x_i + a_i) \rightharpoonup x_i = x_i \rightharpoonup x_i = 0_i$. Hence, we obtain

$$\begin{aligned}\{i \in I : (x_i + a_i) \rightharpoonup x_i > 0_i\} \cap \{i \in I : |y_i| > 0_i\} \subseteq \\ \subseteq \{i \in I : a_i > 0_i\} \cap \{i \in I : |y_i| > 0_i\} = \varnothing,\end{aligned}$$

by (1), since $a \wedge |y| = 0$. Therefore, $((x + a) \rightharpoonup x) \wedge |y| = 0$, and thus $(x + a) \rightharpoonup x \in P^{\perp\perp} = P$. Hence, $x + a = ((x + a) \rightharpoonup x) + x \in P^+ + x$ proving $x + P^+ \subseteq P^+ + x$. One analogously proves the other inclusion.

The implication (iii) $\Rightarrow$ (iv) is obvious and (iv) implies (v) immediately by Lemma 4.

(v) $\Rightarrow$ (ii). Since every prime ideal contains a minimal prime ideal and the intersection of all prime ideals equals to $\{0\}$, so does the intersection of all minimal prime ideals. Thus A is representable. ∎

Theorem 6. *A $DR\ell$-monoid is representable if and only if it satisfies the identities*

$$(x \rightharpoonup y)^+ \wedge (((y \rightharpoonup x)^+ + z) \leftharpoondown z) = 0, \tag{2}$$

$$(x \leftharpoondown y)^+ \wedge ((z + (y \leftharpoondown x)^+) \rightharpoonup z) = 0. \tag{3}$$

Proof. Any linearly ordered $DR\ell$-monoid satisfies (2) and (3) since either $(x \rightharpoonup y)^+ = 0$ or $(y \rightharpoonup x)^+ = 0$ (respectively, either $(x \leftharpoondown y)^+ = 0$ or $(y \leftharpoondown x)^+ = 0$). Therefore the part "only if" is obvious.

Conversely, suppose that the above identities are satisfied by A; then A fulfils also $(*)$ (put $z = 0$). In view of Theorem 5, it suffices to prove that $x^\perp$ is normal for all $x \in A^+$.

Let $y \in (x^\perp)^+$, that is, $y \wedge x = 0$. Observe that in this case

$$x = x \rightharpoonup (y \wedge x) = (x \rightharpoonup y) \vee (x \rightharpoonup x) = (x \rightharpoonup y) \vee 0 = (x \rightharpoonup y)^+,$$

and similarly $y = (y \rightharpoonup x)^+$. Hence, by (2),

$$x \wedge ((y + z) \leftharpoondown z) = (x \rightharpoonup y)^+ \wedge (((y \rightharpoonup x)^+ + z) \leftharpoondown z) = 0;$$

thus $(y + z) \leftharpoondown z \in (x^\perp)^+$. Further, $y + z \geq z$ implies $y + z = (y + z) \vee z = z + ((y + z) \leftharpoondown z) \in z + (x^\perp)^+$ which shows $(x^\perp)^+ + z \subseteq z + (x^\perp)^+$. The other inclusion follows similarly by (3). ∎

Corollary 7. *A $DR\ell$-monoid is representable if and only if it satisfies the identities*

$$(x \rightharpoonup y) \wedge (((y \rightharpoonup x) + z) \leftharpoondown z) \leq 0,$$

$$(x \leftharpoondown y) \wedge ((z + (y \leftharpoondown x)) \rightharpoonup z) \leq 0.$$

Proof. One readily sees that

$$(x \rightharpoonup y)^+ \wedge (((y \rightharpoonup x)^+ + z) \leftharpoondown z) = [(x \rightharpoonup y) \wedge (((y \rightharpoonup x) + z) \leftharpoondown z)]^+,$$

$$(x \leftharpoondown y)^+ \wedge ((z + (y \leftharpoondown x)^+) \rightharpoonup z) = [(x \leftharpoondown y) \wedge ((z + (y \leftharpoondown x)) \rightharpoonup z)]^+.$$

∎

Corollary 8. *The class of all representable $DR\ell$-monoids is a proper subvariety of the variety of all $DR\ell$-monoids.*

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