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CLASSIFICATION SYSTEMS AND THEIR LATTICE *

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Abstract

We define and study classification systems in an arbitrary CJ-generated complete lattice L. Introducing a partial order among the classification systems of L, we obtain a complete lattice denoted by Cls(L). By using the elements of the classification systems, another lattice is also constructed: the box lattice $\mathcal{B}(L)$ of L. We show that $\mathcal{B}(L)$ is an atomistic complete lattice, moreover Cls(L)=Cls(B(L)). If $\mathcal{B}(L)$ is a pseudocomplemented lattice, then every classification system of L is independent and Cls(L) is a partition lattice.

Keywords: concept lattice, CJ-generated complete lattice, atomistic complete lattice, (independent) classification system, classification lattice, box lattice.

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1. INTRODUCTION

The notion of the classification system is related to an application of Concept Lattices to one of the main problems in Group Technology, namely, to classify some technological objects on the basis of their common properties (attributes) (see [3]). The dual of this notion was introduced first in the literature by R. Wille ([7]).

Although classification systems were defined in concept lattices, they can be introduced even in an arbitrary complete lattice (see [4]).

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The basic notions and results are presented in Section 2. In Section 3, classification systems of a CJ-generated complete lattice L are studied. We show that they can be ordered in a natural way. In this manner we obtain a complete lattice, called the classification lattice of the lattice L. This result is proved in Section 4. Using the notion of the box lattice, it is also shown that the classification lattice of any lattice can be represented as a classification lattice of an atomistic lattice. In Section 5 it is proved that a classification lattice can be considered as a generalization of a partition lattice. In Section 6 by using independent classification systems we determine the box lattice and the classification lattice of a CJ-generated complete pseudocomplemented lattice.

2. Preliminaries

Given a set G of objects and a set M of attributes, a binary relation $I \subseteqq G \times M$ is defined as follows:

$$(g, m) \in I \text{ if and only if the object } g \in G \text{ has the attribute } m \in M.$$

The triple (G, M, I) is called a *formal context* in mathematical literature (see, e.g., [2]). By defining

$$A' = \{m \in M \mid (g, m) \in I, \text{ for all } g \in A\},$$

$$B' = \{g \in G \mid (g, m) \in I, \text{ for all } m \in B\}$$

for all subsets $A \subseteqq G$ and $B \subseteqq M$, we establish a *Galois connection* between G and M. The pairs (A, B) with $A' = B$ and $B' = A$ are called the formal concepts of the context (G, M, I). The formal concepts of (G, M, I) together with the partial order defined by

$$(A_1, B_1) \leqq (A_2, B_2) \Leftrightarrow A_1 \subseteqq A_2 \text{ (or equivalently } B_2 \subseteqq B_1)$$

form a complete lattice $\mathcal{L}(G, M, I)$ which is called the *concept lattice of the context* (G, M, I). The infimum and the supremum in the lattice $\mathcal{L}(G, M, I)$ are given by

$$\bigwedge_{i\in I}(A_i,B_i)=\left(\bigcap_{i\in I}A_i,\left(\bigcup_{i\in I}B_i\right)''\right),$$

$$\bigvee_{i\in I}(A_i,B_i)=\left(\left(\bigcup_{i\in I}A_i\right)'',\bigcap_{i\in I}B_i\right).$$

Remark 1.1. We have $A'=A'''$ for each $A\subseteqq G$. Therefore, if $A=A''$, then the pair (A,A') is a formal concept of the context (G,M,I).

For any $g\in G$ we define the concept $\gamma(g)=(\{g\}'',\{g\}')$. Then for any concept $(A,B)\in\mathcal{L}(G,M,I)$ we have $(A,B)=\bigvee_{g\in A}\gamma(g)$ (see [2]). A nonzero element p of a lattice L is called *completely join-irreducible* if for any system of elements $x_i\in L,\ i\in I$ the equality $p=\bigvee\{x_i\mid i\in I\}$ implies $p=x_{i_0}$ for some $i_0\in I$. If any nonzero element of L is a join of completely join-irreducible elements, then L is called a *CJ-generated lattice.* A context (G,M,I) is called *row-reduced* if every $\gamma(g)$, $g\in G$ is completely join-irreducible. Therefore the concept lattice of a row-reduced context is a CJ-generated lattice. In this case the $\mathbf{0}$ element of the lattice $\mathcal{L}(G,M,I)$ is the concept $(\varnothing,M)$.

A classification of the elements of G means a partition $\pi=\{G_j,\,j\in J\}$ of G, where any block G_j of π is characterized by the common attributes of its objects, i.e. where $G_j''=G_j$, for all $j\in J$. In this case any $a_j=(G_j,G_j')$ is a concept of (G,M,I), i.e. $a_i\in\mathcal{L}(G,M,I)$.

Definition 2.2. Let L be a complete lattice. A set $S=\{a_j\mid j\in J\}$, $J\neq\varnothing$ of nonzero elements of L is called a *classification system* of L if the following conditions are satisfied:

(1) $\qquad a_i\wedge a_j=\mathbf{0},\quad$ for all $\ i\neq j$,

(2) $\qquad x=\bigvee\limits_{j\in J}(x\wedge a_j)$, for all $x\in L$.

Clearly, (2) implies $\mathbf{1}=\bigvee\limits_{j\in J}a_j$ and it is equivalent to

(2′) $\qquad x\leqq\bigvee\limits_{j\in J}(x\wedge a_j)$, for all $x\in L\smallsetminus\{\mathbf{0}\}$.

If $S = \{\mathbf{1}\}$, then we say that the classification system S is *trivial.*

The above definition was motivated by the following result of [3]:

Theorem 2.3. *If (G, M, I) is a row-reduced context, then to any classification $\pi = \{G_j,\ j \in J\}$ of the elements of G corresponds a classification system $a_j = (G_j, G'_j),\ j \in J$ of the lattice $\mathcal{L}(G, M, I)$. Conversely, any classification system of the lattice $\mathcal{L}(G, M, I)$ induces a classification of the elements of G.*

To make our paper self-contained, we present here the proof of the above theorem.

Proof. As π is a partition of G, we have $G_j \neq \varnothing$, and hence $a_j = (G_j, G'_j) \neq \mathbf{0}$. It is also clear that $G_i \cap G_j = \varnothing$ implies $a_i \wedge a_j = \mathbf{0}$ (whenever $i \neq j$). Thus (1) is satisfied.

Now take $x = (A, B) \in \mathcal{L}(G, M, I)$, $x \neq 0$. Then $x = \bigvee_{g \in A} \gamma(g)$. As π is a partition of G, for each $g \in A$ there exists an $j_g \in J$ such that $g \in G_{j_g}$. Hence we get $\gamma(g) \leqq (G_{j_g}, G'_{j_g}) = a_{j_g}$ and $\gamma(g) \leqq x$ implies $\gamma(g) \leqq x \wedge a_{j_g}$. Thus we obtain $x = \bigvee_{g \in A} \gamma(g) \leqq \bigvee_{j \in J} (x \wedge a_j)$, i.e. $(2')$.

Conversely, assume that the system $a_j \in \mathcal{L}(G, M, I)$, $j \in J$ satisfies the conditions (1) and (2). Clearly, any a_j is of the form $a_j = (A_j, B_j)$, where $A_j \subseteqq G$, $B_j \subseteqq M$ and $A'_j = B_j$ and $B'_j = A_j$. First, we prove that the sets A_j, $j \in J$ form a partition of G.

Indeed, $a_j \neq \mathbf{0}$ gives $A_j \neq \varnothing$, for all $j \in J$. It is also obvious that $a_i \wedge a_j = \mathbf{0}$ implies $A_i \cap A_j = \varnothing$, whenever $i \neq j$.

Let $g \in G$. Since $\gamma(g)$ is completely join-irreducible, $\gamma(g) = \bigvee_{j \in J} (\gamma(g) \wedge a_j)$ implies $\gamma(g) = \gamma(g) \wedge a_{j_0}$ for some $j_0 \in J$. Hence $\gamma(g) \leqq a_{j_0}$, and this gives $\{g\}'' \subseteqq A_{j_0}$. As $g \in \{g\}''$, we get $g \in \bigcup_{j \in J} A_j$, proving $G = \bigcup_{j \in J} A_j$.

As $A'_j = B_j$ and $B'_j = A_j$ implies $A_j = A''_j$, $\{A_j,\ j \in J\}$ is a classification of the elements of G. ■

3. Classification systems in CJ-generated complete lattices

In what follows let L be a CJ-generated complete lattice. The set of all completely join-irreducible elements of L is denoted by $J(L)$. For $a \in L$ let $J(a) = \{p \in J(L) \mid p \leq a\}$ and set $\bigvee \varnothing = \mathbf{0}$. Now, $a = \bigvee J(a)$ and for any system of elements $a_i \in L$, $i \in I$ we have:

(3) $$J\left(\bigwedge_{i\in I} a_i\right) = \bigcap_{i\in I} J(a_i) \text{ and } \bigcup_{i\in I} J(a_i) \subseteqq J\left(\bigvee_{i\in I} a_i\right).$$

If every $p \in J(L)$ is an atom in L, then L is called *atomistic.*
A set $A \subseteqq J(L)$ is called $\vee$-*closed* if for any $p \in J(L)$ $p \leq \bigvee A$ implies $p \in A$. Clearly, A is $\vee$-closed if and only if $A = J(a)$, where $a = \bigvee A$.

Remark 3.1. By using (3) it is easy to see that any nonempty intersection of $\vee$-closed sets of $J(L)$ is a $\vee$-closed set again.

Definition 3.2. A partition $\pi = \{A_i,\ i \in I\}$ of $J(L)$ is called $\vee$-*closed* if any block A_i of π is a $\vee$-closed set.

Proposition 3.3. (i) *If $S = \{a_i \mid i \in I\}$ is a classification system of L, then $\pi_S = \{J(a_i), i \in I\}$ is a $\vee$-closed partition of $J(L)$.*

(ii) *If $\pi = \{A_i, i \in I\}$ is a $\vee$-closed partition of $J(L)$ and $a_i = \bigvee A_i$, then $S_\pi = \{a_i \mid i \in I\}$ is a classification system of L with $J(a_i) = A_i$.*

Proof. (i) $a_i \neq \mathbf{0}$ implies $J(a_i) \neq \varnothing$, and for all $i \neq j$ $a_i \wedge a_j = \mathbf{0}$ gives $A_i \cap A_j = \varnothing$. Let $p \in J(L)$, then $p = \bigvee_{i\in I}(p \wedge a_i)$ implies $p = p \wedge a_{i_0}$, i.e. $p \in J(a_{i_0})$ for some $i_0 \in I$. Hence $J(L) \subseteqq \bigcup_{i\in I} J(a_i)$. As the reversed inclusion is obvious $\pi_S = \{J(a_i), i \in I\}$ is a partition of $J(L)$. Since any set $J(a_i)$ is $\vee$-closed, π_S is a $\vee$-closed partition of $J(L)$.

(ii) $A_i \neq \varnothing$ gives $a_i \neq \mathbf{0}$. Let $i, j \in I$, $i \neq j$. For the $\vee$-closed sets A_i and A_j we have $A_i = J(a_i)$ and $A_j = J(a_j)$. Hence we get $J(a_i \wedge a_j) = J(a_i) \cap J(a_j) = A_i \cap A_j = \varnothing$, and this gives $a_i \wedge a_j = \mathbf{0}$.

Let $x \in L \smallsetminus \{\mathbf{0}\}$ and take any $p \in J(x)$. As $\{A_i, i \in I\}$ is a partition of $J(L)$, there is an $i_p \in I$ such that $p \in A_{i_p}$. Thus we get $p \leqq x \wedge (\bigvee A_{i_p}) = x \wedge a_{i_p}$, and this implies $x = \bigvee\{p \mid p \in J(x)\} \leqq \bigvee_{i\in I}(x \wedge a_i)$.

As we have proved (1) and (2′), $S_\pi = \{a_i \mid i \in I\}$ is a classification system of L. Since $J(a_i) = A_i, i \in I$, the proof is complete. ■

Remark 3.4. It is implicit in the proof that $S_{\pi_S} = S$ and $\pi_{S_\pi} = \pi$.

Corollary 3.5. *If L is a CJ-generated lattice, then any classification system $S = \{a_i \mid i \in I\}$ of L is determined by the partition π_S, induced by S on $J(L)$.*

Proof. Let S and T be two classification systems of the lattice L and suppose that $\pi_S = \pi_T$. Then, according to Remark 3.4, we obtain $S = S_{\pi_S} = S_{\pi_T} = T$. ■

4. The classification lattice and the box lattice of a CJ-generated complete lattice

Definition 4.1. Let L be a CJ-generated complete lattice and let S_1 and S_2 be two classification systems of L. We say that the system S_1 is *finer* then S_2 and we write $S_1 \leq S_2$, if the partition π_{S_1} induced by S_1 refines the partition π_{S_2} induced by S_2, i.e. if $\pi_{S_1} \leqq \pi_{S_2}$.

Let $\mathrm{Cls}(L)$ denote the set of all classification systems of the lattice L. It is easy to see that $\leq$ is a partial order on $\mathrm{Cls}(L)$. Indeed, $\leq$ is reflexive and transitive by definition. Let S_1, S_2 with $S_1 \leq S_2$ and $S_2 \leq S_1$. Then $\pi_{S_1} \leqq \pi_{S_2}$ and $\pi_{S_2} \leqq \pi_{S_1}$ implies $\pi_{S_1} = \pi_{S_2}$. Now applying Corollary 3.5, we obtain $S_1 = S_2$.

Theorem 4.2. $(\mathrm{Cls}(L), \leq)$ *is a complete lattice.*

Proof. As the trivial classification $S = \{\mathbf{1}\}$ is the greatest element of $(\mathrm{Cls}(L), \leq)$, to prove that $(\mathrm{Cls}(L), \leq)$ is a complete lattice it is enough to show that for any nonempty set $S_k \in \mathrm{Cls}(L)$, $k \in K$ the infimum $\bigwedge_{k\in K} S_k$ exists in $(\mathrm{Cls}(L), \leq)$. Let π_{S_k} stand for the partitions induced by $S_k \in \mathrm{Cls}(L)$, $k \in K$. As the partitions of a set form a complete lattice, we can consider the partition $\bigwedge_{k\in K} \pi_{S_k}$ of $J(L)$. Since any block of $\bigwedge_{k\in K} \pi_{S_k}$ is an intersection of some blocks of the $\vee$-closed partitions π_{S_k}, all the blocks of $\bigwedge_{k\in K} \pi_{S_k}$ are $\vee$-closed. Hence, in view of Proposition 3.3 (ii) and Remark 3.4, there exists a classification system $S = \{a_i \mid i \in I\} \subseteqq L$ such that $\pi_S = \bigwedge_{k\in K} \pi_{S_k}$. We claim $S = \bigwedge_{k\in K} S_k$.

Indeed, $\pi_S \leqq \pi_{S_k}, k \in K$ implies $S \leq S_k, k \in K$. Now take a $T \in \mathrm{Cls}(L)$ such that $T \leq S_k, k \in K$. Then $\pi_T \leqq \pi_{S_k}, k \in K$ implies $\pi_T \leqq \bigwedge_{k\in K} \pi_{S_k} = \pi_S$, whence we get $T \leq S$. Thus $S = \bigwedge_{k\in K} S_k$. ■

Remark 4.3. (i) $(\mathrm{Cls}(L), \leq)$ or $\mathrm{Cls}(L)$ for short, is called *the classification lattice* of the lattice L. Observe, that in the proof of Theorem 4.2 we have also established that $\bigwedge_{k\in K} \pi_{S_k}$ is induced by $\bigwedge_{k\in K} S_k$.

(ii) Notice that the $\mathbf{0}$-element of the lattice $\mathrm{Cls}(L)$, i.e. the finest classification system of L, is the same as $S_0 = \bigwedge\{S \mid S \in \mathrm{Cls}(L)\}$.

Definition 4.4. The $\mathbf{0}$ of a lattice L and any element of a classification system of L is called a *box element* of L. The set of the box elements of L is denoted by $\mathcal{B}(L)$.

Let $(\mathcal{B}(L), \leqq)$ denote the partially ordered set which is obtained by restricting the partial order of L to $\mathcal{B}(L)$.

Theorem 4.5. $(\mathcal{B}(L), \leq)$ *is an atomistic complete lattice whose atoms are the elements of the finest classification system* S_0 *of* L.

Proof. As $\mathbf{1}$ is an element of the trivial classification system $S = \{\mathbf{1}\}$, we have $\mathbf{1} \in \mathcal{B}(L)$. Clearly, $\mathbf{1}$ is the greatest element of $\mathcal{B}(L)$. Therefore, to prove that $\mathcal{B}(L)$ is a complete lattice, it is enough to show that the infimum of any nonempty system $b_k \in \mathcal{B}(L)$, $k \in K$ exists in $(\mathcal{B}(L), \leq)$. Since the required infimum is $\bigwedge_{k \in K} b_k$ whenever $\bigwedge_{k \in K} b_k \in \mathcal{B}(L)$, we shall proceed by proving $\bigwedge_{k \in K} b_k \in \mathcal{B}(L)$.

As $\mathbf{0} \in \mathcal{B}(L)$, if $\bigwedge_{k \in K} b_k = \mathbf{0}$, then we are done. Suppose that $\bigwedge_{k \in K} b_k \neq \mathbf{0}$: Then we have $b_k \neq \mathbf{0}$ for all $k \in K$. In view of Definition 4.4, there exists a classification system S_k with $b_k \in S_k$, for each $k \in K$. Let us consider now the partitions π_{S_k} induced by the systems S_k, $k \in K$ and let $\bigwedge_{k \in K} S_k = \{a_i \mid i \in I\}$. Since $J\left(\bigwedge_{k \in K} b_k\right) = \bigcap_{k \in K} J(b_k)$ is a block of the partition $\bigwedge_{k \in K} \pi_{S_k}$ induced by $\bigwedge_{k \in K} S_k$, there exists an element a_{i_0} ($i_0 \in I$) of $\bigwedge_{k \in K} S_k$ such that $J(a_{i_0}) = J\left(\bigwedge_{k \in K} b_k\right)$. Hence we obtain $\bigwedge_{k \in K} b_k = a_{i_0} \in \mathcal{B}(L)$ proving that $(\mathcal{B}(L), \leqq)$ is a complete lattice.

Now we prove that $(\mathcal{B}(L), \leqq)$ is atomistic. Let $\sqcup$ stand for the join operation in $(\mathcal{B}(L), \leqq)$. Then for any $b_k \in \mathcal{B}(L)$, $k \in K$ we have:

$$\bigsqcup_{k \in K} b_k = \bigwedge \{b \in \mathcal{B}(L) \mid b_k \leqq b, \text{ for all } k \in K\} \geqq \bigvee_{k \in K} b_k. \tag{4}$$

Let $S_0 = \{\alpha_i \mid i \in I_0\}$ be the finest classification system of L. Observe, that there is no element $b \in L$ which belongs to some $S \in \mathrm{Cls}(L)$ and satisfies $\mathbf{0} < b < \alpha_i$ for some $i \in I_0$. (Otherwise, $\pi_S \wedge \pi_{S_0} < \pi_{S_0}$ would imply $S \wedge S_0 < S_0$ - a contradiction). Therefore, any α_i is an atom in $(\mathcal{B}(L), \leqq)$. We prove that any $b \in \mathcal{B}(L)$ is a "$\sqcup$-join" of some subset of $\{\alpha_i \mid i \in I_0\}$.

Since $b \wedge \alpha_i \in \mathcal{B}(L)$ and since any α_i is an atom in $\mathcal{B}(L)$, we have either $b \wedge \alpha_i = \mathbf{0}$ or $b \wedge \alpha_i = \alpha_i$ for each $i \in I_0$. Let $M = \{i \in I_0 \mid b \wedge \alpha_i = \alpha_i\}$. Then $\alpha_i \leqq b$, $i \in M$ and hence we get $\bigsqcup_{i \in M} \alpha_i \leqq b$. As $\{\alpha_i \mid i \in I_0\}$ is a classification system in L, by using (4) we obtain:

$$b = \bigvee_{i \in I_0} (b \wedge \alpha_i) = \bigvee_{i \in M} \alpha_i \leqq \bigsqcup_{i \in M} \alpha_i \tag{5}$$

Thus we get $b = \bigsqcup_{i \in M} \alpha_i$. This relation also implies that any atom of $\mathcal{B}(L)$ is an element of S_0 and the proof is completed. ∎

Remark 4.6. We shall use the short notation $\mathcal{B}(L)$ for the box lattice of L in what follows. It is implicit in the above proof that the $\mathbf{0}$-element and the meet operation coincide in L and in $\mathcal{B}(L)$. In the proof it is also shown that for any system $b_k \in \mathcal{B}(L)$, $k \in K$ we have $\bigvee_{k\in K} b_k \leqq \bigsqcup_{k\in K} b_k$ and any $b \in \mathcal{B}(L)$ is of the form $b = \bigvee_{i\in M} \alpha_i$ with some $M \subseteqq I_0$.

Theorem 4.7. *For any CJ-generated complete lattice L we have* $\mathrm{Cls}(L) = \mathrm{Cls}(\mathcal{B}(L))$.

Proof. Let $S = \{a_i \mid i \in I\}$ be a classification system of the lattice L. We prove that S is also a classification system of $\mathcal{B}(L)$.

Indeed, we have $a_i \in \mathcal{B}(L) \setminus \{\mathbf{0}\}$ for all $i \in I$. As the $\mathbf{0}$-element and the $\wedge$ operation is the same in L and $\mathcal{B}(L)$, for all $i \neq j$ the equality $a_i \wedge a_j = \mathbf{0}$ holds in the lattice $\mathcal{B}(L)$ also. Now take a $b \in \mathcal{B}(L)$. Then, in view of Remark 4.6, we get $b = \bigvee_{i\in I}(b \wedge a_i) \leqq \bigsqcup_{i\in I}(b \wedge a_i) \leqq b$. Therefore, $b = \bigsqcup_{i\in I}(b \wedge a_i)$ also holds in $\mathcal{B}(L)$. Thus we obtain $\mathrm{Cls}(L) \subseteqq \mathrm{Cls}(\mathcal{B}(L))$.

Conversely, take a system $S = \{b_k \mid k \in K\} \in \mathrm{Cls}(\mathcal{B}(L))$. Clearly, $b_i \wedge b_j = 0$, for all $i, j \in K$, $i \neq j$. Take an $x \in L \setminus \{0\}$. Now we show that S satisfies the inequality $(2')$ in the lattice L.

Let $p \in J(x)$ and consider the partition $\{J(\alpha_i) \mid i \in I_0\}$ induced by the classification system $S_0 = \{\alpha_i \mid i \in I_0\}$. Obviously, we have $p \in J(\alpha_{i_p})$ for some $i_p \in I_0$, i.e. we get $p \leqq \alpha_{i_p}$. Since $\alpha_{i_p} \in \mathcal{B}(L)$ and since S is a classification system of $\mathcal{B}(L)$, we obtain $\alpha_{i_p} = \bigsqcup_{k\in K}(\alpha_{i_p} \wedge b_k)$. As α_{i_p} is an atom of $\mathcal{B}(L)$, there exists a $k_p \in K$ such that $\alpha_{i_p} = \alpha_{i_p} \wedge b_{k_p}$. Hence $p \leqq \alpha_{i_p} \leqq b_{k_p}$ and this implies $p \leqq x \wedge b_{k_p} \leqq \bigvee_{k\in K}(x \wedge b_k)$.

Thus we obtain $x = \bigvee\{p \mid p \in J(x)\} \leqq \bigvee_{k\in K}(x \wedge b_k)$, i.e. $(2')$. Therefore, $S = \{b_k \mid k \in K\}$ is a classification system of L. Hence $\mathrm{Cls}(\mathcal{B}(L)) \subseteqq \mathrm{Cls}(L)$, completing the proof. ∎

We say that $\mathcal{L}$ is a *classification lattice*, if $\mathcal{L}$ is isomorphic to some $\mathrm{Cls}(L)$ (where L is a CJ-generated complete lattice). Now, as a consequence of Theorem 4.7 and Theorem 4.5, we can formulate

Corollary 4.8. *Any classification lattice can be represented as the classification lattice of an atomistic complete lattice.*

5. Classification lattices of atomistic complete lattices

Let L be an atomistic complete lattice and denote by $A(L)$ the set of its atoms. For $x \in L$ let $A(x) = \{a \in A(L) \mid a \leq x\}$. As any atomistic lattice is a CJ-generated lattice with $J(L) = A(L)$, $S = \{a_i \mid i \in I\} \subseteqq L$ is a classification system of L if and only if $\pi_S = \{A(a_i), i \in I\}$ is a $\vee$-closed partition of $A(L)$.

Further, observe that for any element $x \in L \setminus \{\mathbf{0}\}$ the set

$S_x = \{x\} \cup \{a \in A(L) \mid a \wedge x = \mathbf{0}\}$ is a classification system of L.

Indeed, let us consider the sets $\{a\} \subseteqq A(L)$ with $a \wedge x = \mathbf{0}$ and the set $A(x)$. Since these sets are nonempty and pairwise disjoint and since their union is $A(L)$, they form a partition of $A(L)$, denoted by π_x. As all these sets are $\vee$-closed subsets of $A(L)$, π_x is a $\vee$-closed partition. Now, in virtue of Proposition 3.3 (ii), the relations $x = \bigvee A(x)$ and $a = \bigvee\{a\}$ imply that S_x is a classification system of L with $\pi_{S_x} = \pi_x$.

Therefore, any $x \in L$ is a box element of L and hence $\mathcal{B}(L) = L$. Moreover, the meet operation is the same in L and $\mathcal{B}(L)$ (see Remark 4.6) and for any $a, b \in \mathcal{B}(L)$ we have:

$$a \sqcup b = \bigwedge\{x \in \mathcal{B}(L) \mid a \leqq x,\ b \leqq x\} = \bigwedge\{x \in L \mid a \leqq x,\ b \leqq x\} = a \vee b.$$

As a consequence, we obtain the following

Proposition 5.1. *If L is an atomistic complete lattice, then its box lattice $\mathcal{B}(L)$ coincides with L.*

Corollary 5.2. *In any atomistic complete lattice L the system $A(L)$ of all atoms of L is the finest classification system of L.*

Proof. By Theorem 4.5 the finest classification system of L is $S_0 = \{\alpha \in L \mid \alpha \text{ is an atom in } \mathcal{B}(L)\}$. Now $\mathcal{B}(L) = L$ gives $S_0 = A(L)$. ∎

Remark 5.3. Clearly, $S_x < S_y \Leftrightarrow x < y$ and $S_x = A(L) \Leftrightarrow x \in A(L)$. Moreover, if $x, y \notin A(L)$, then $S_x = S_y \Rightarrow x = y$. Further, denote by $\veebar$ the join operation in $\mathrm{Cls}(L)$ and let $S = \{a_i \mid i \in I\}$ be a classification system of L. Then it is easy to check that $S = \underline{\bigvee}\{S_{a_i} \mid i \in I\}$.

We say that a lattice L satisfies the *local Birkhoff condition*, if for any atoms $a, b \in L$ we have $a, b \prec a \vee b$. (We note that this condition is satisfied by any semimodular and any $\mathbf{0}$-modular lattice - see, e.g., [6].) Let L be a complete lattice and for any $p \in L$ let $\widetilde{p} = \bigvee\{x \in L \mid x < p\}$. If $p \in J(L)$, then it can be easily seen that $\widetilde{p}$ is the unique element of L satisfying $\widetilde{p} \prec p$. Hence $\widetilde{p} = \mathbf{0}$ if and only if p is an atom in L. Finally, let $\mathrm{Part}(A)$ stand for the partition lattice of a set A.

Proposition 5.4. *If L is an atomistic complete lattice satisfying the local Birkhoff condition, then any completely join-irreducible element of* $\mathrm{Cls}(L)$ *is an atom in* $\mathrm{Cls}(L)$.

Proof. Let S be a completely join-irreducible element of $\mathrm{Cls}(L)$. Then, in view of Remark 5.3, there is an element $d \in L \setminus \{\mathbf{0}\}$, $d \notin A(L)$ such that $S = S_d$. Let us consider now $\widetilde{S} = \underline{\bigvee}\{T \in \mathrm{Cls}(L) \mid T < S\}$. As $A(L)$ is the $\mathbf{0}$-element of $\mathrm{Cls}(L)$, we have to prove only $\widetilde{S} = A(L)$.

In contrary, assume that $\widetilde{S}$ contains an element $c \notin A(L)$. As $S_c \leq \widetilde{S} < S_d$, we get $\mathbf{0} < c < d$, according to Remark 5.3. Then there are $a, b \in A(d)$ with $a < c$ and $b \wedge c = \mathbf{0}$, and $a \vee b \leqq d$ implies $S_{a\vee b} \leq S$. Since $a \vee b \nleqq c$ and since $(a \vee b) \wedge c \geqq a > \mathbf{0}$, there is no $x \in \widetilde{S}$ with $a \vee b \leqq x$. (Otherwise $a \vee b \leqq x$ and $x, c \in \widetilde{S}$, $x \neq c$ would imply $\mathbf{0} = x \wedge c \geqq (a \vee b) \wedge c > \mathbf{0}$, a contradiction.) Hence $S_{a\vee b}$ does not satisfy $S_{a\vee b} \leq \widetilde{S}$. Therefore, $S_{a\vee b} \leq S$ implies $S_{a\vee b} = S = S_d$. As $a \vee b$, $d \notin A(L)$, in view of Remark 5.3, we get $a \vee b = d$. Finally, by using the local Birkhoff condition, we obtain $a \prec d$, contrary to $a < c < d$. ∎

Remark 5.5. It is implicit in the above proof, that any atom of $\mathrm{Cls}(L)$ is of the form $S = S_{a\vee b}$, with $a, b \in A(L)$, moreover, if L satisfies the local Birkhoff condition, then any such an $S_{a\vee b}$ is an atom in $\mathrm{Cls}(L)$.

Corollary 5.6. *Let L be a complete CJ-generated lattice. If $\mathcal{B}(L)$ is finite and satisfies the local Birkhoff condition, then* $\mathrm{Cls}(L)$ *is a finite atomistic lattice.*

Proof. Since $\mathcal{B}(L)$ is finite, $A(\mathcal{B}(L))$ and $\mathrm{Part}(A(\mathcal{B}(L)))$ are also finite. As $S \mapsto \pi_s$ maps injectively $\mathrm{Cls}(\mathcal{B}(L))$ in $\mathrm{Part}(A(\mathcal{B}(L)))$, $\mathrm{Cls}(\mathcal{B}(L))$ is finite, too. Thus $\mathrm{Cls}(L) = \mathrm{Cls}(\mathcal{B}(L))$ is a finite lattice. As any element of a finite lattice is a join of completely join-irreducible elements of it, by applying Proposition 5.4, we obtain that $\mathrm{Cls}(L)$ is atomistic. ∎

Let L_1, L_2 be two lattices, the map $\varphi\colon L_1 \longrightarrow L_2$ is a $\wedge$-*homomorphism* if for any $a, b \in L_1$ we have $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$. Clearly, any $\wedge$-homomorphism is order preserving. We recall (see e.g. [5]) that L_1 and L_2 are isomorphic iff there are order preserving maps $\varphi\colon L_1 \longrightarrow L_2$ and $\psi\colon L_2 \longrightarrow L_1$, such that φ and ψ are inverses of each other. We show that any bijective $\wedge$-homomorphism φ is a lattice isomorphism:

Indeed, as φ is order preserving, it is enough to prove that its inverse φ^{-1} is also order preserving, that is $\varphi(a) \leqq \varphi(b) \implies a \leqq b$. Assume that $\varphi(a) \leqq \varphi(b)$. Then $\varphi(a) = \varphi(a) \wedge \varphi(b) = \varphi(a \wedge b)$ and hence the injectivity of φ implies $a = a \wedge b$, i.e. $a \leqq b$.

Lemma 5.7. *Let L be an atomistic complete Boolean lattice. Then*

(i) *any partition of the set $A(L)$ is $\vee$-closed*;

(ii) *the classification lattice of L is isomorphic to* $\mathrm{Part}(A(L))$.

Proof. (i) Let $\pi = \{A_i, i \in I\}$ be a partition of $A(L)$ and take an $x \in A(L)$ such that $x \leqq \bigvee A_i$ for some $i \in I$. We prove $x \in A_i$.

Suppose that $x \notin A_i$, then we have $x \wedge a = \mathbf{0}$ for each $a \in A_i$. As any complete Boolean lattice is infinitely distributive, we get $x = x \wedge (\bigvee A_i) = \bigvee_{a \in A_i}(x \wedge a) = \mathbf{0}$ - a contradiction.

This shows that $\pi = \{A_i, i \in I\}$ is a $\vee$-closed partition.

(ii) Let us consider the mapping

$$\varphi\colon \mathrm{Cls}(L) \longrightarrow \mathrm{Part}(A(L)),\ \varphi(S) = \pi_S.$$

We prove that φ is a isomorphism, by showing that it is a bijective $\wedge$-homomorphism.

Take any $S_1, S_2 \in \mathrm{Cls}(L)$. In view of Remark 4.3 (i), the classification system $S_1 \wedge S_2$ induces the partition $\pi_{S_1} \wedge \pi_{S_2}$ of $A(L)$. Therefore, we have $\varphi(S_1 \wedge S_2) = \varphi(S_1) \wedge \varphi(S_2)$ and hence φ is a $\wedge$- homomorphism.

Further, assume that $\varphi(S_1) = \varphi(S_2)$ for some $S_1, S_2 \in \mathrm{Cls}(L)$. In view of Corollary 3.5, $\pi_{S_1} = \pi_{S_2}$ implies $S_1 = S_2$. Hence φ is injective.

Take a partition $\pi = \{A_i, i \in I\}$ of $A(L)$ and let $a_i = \bigvee A_i$, $i \in I$. As π is $\vee$-closed according to (i), Proposition 3.3 (ii) gives that $S_\pi = \{a_i \mid i \in I\}$ is a classification system in L. Using Remark 3.4, we get that the partition induced by S_π coincides with π, i.e. $\varphi(S_\pi) = \pi$. Thus φ is surjective and the proof is completed. ∎

Combining Theorem 4.5, Theorem 4.7 and this result, we obtain

Corollary 5.8. *Let L be a CJ-generated complete lattice. If $\mathcal{B}(L)$ is a Boolean lattice, then* $\mathrm{Cls}(L)$ *is isomorphic to a partition lattice.*

Finally, we show that the notion of the classification lattice can be considered as a generalization of the notion of the partition lattice:

Proposition 5.9. *Any partition lattice can be represented as the classification lattice of an atomistic complete Boolean lattice.*

Proof. Consider the partition lattice of an arbitrary nonempty set A. Since the power-set lattice $\mathcal{P}(A)$ of the set A is an atomistic complete Boolean lattice and since the atoms of this lattice are the sets $\{a\}$, $a \in A$, Lemma 5.7 gives $\mathrm{Cls}(\mathcal{P}(A)) \cong \mathrm{Part}(\{\{a\} \mid a \in A\}) \cong \mathrm{Part}(A)$. ∎

6. Independent classification systems and pseudocomplemented lattices

Let L be a complete lattice. A classification system $S = \{a_i \mid i \in I\}$ of L is called *independent* if $a_j \wedge (\bigvee_{i \in I \setminus \{j\}} a_i) = \mathbf{0}$ holds for all $j \in I$.

The following lemma from [4] will be useful in our proofs.

Lemma 6.1. *Let $S = \{a_i \mid i \in I\}$ be an independent classification system of a complete lattice L. Then for any $K \subseteqq I$, $K \neq \varnothing$ and $b = \bigvee_{i \in K} a_i$, $S^* = \{a_i \mid i \in I \setminus K\} \cup \{b\}$ is also a classification system.*

Corollary 6.2. *Let $S = \{a_i \mid i \in I\}$ be an independent classification system and $K \subseteqq I$. Then*:

(i) $\bigvee\limits_{i \in K} a_i \in \mathcal{B}(L)$;

(ii) $a_j \leqq \bigvee\limits_{i \in K} a_i$ *implies* $j \in K$;

(iii) *for any* $J \subseteqq I$, *we have* $\left(\bigvee\limits_{i \in J} a_i\right) \wedge \left(\bigvee\limits_{i \in K} a_i\right) = \bigvee\limits_{i \in J \cap K} a_i$.

Proof. As $\bigvee \varnothing = \mathbf{0} \in \mathcal{B}(L)$, (i) is an easy consequence of Lemma 6.1.

(ii) Let $b = \bigvee\limits_{i \in K} a_i$ and suppose that $j \in I \setminus K$. Then Lemma 6.1 implies $a_j \wedge b = \mathbf{0}$, contrary to $\mathbf{0} < a_j \leqq b$.

(iii) Let $b_1 = \bigvee\limits_{i \in J} a_i$, $b_2 = \bigvee\limits_{i \in K} a_i$, $c = b_1 \wedge b_2$ and $I_c = \{i \in I \mid a_i \wedge c \neq \mathbf{0}\}$. Clearly, $J \cap K \subseteqq I_c$. Further, Lemma 6.1 gives $a_i \wedge b_1 = \mathbf{0}$ for any $i \in I \setminus J$ and $a_i \wedge b_2 = \mathbf{0}$ for any $i \in I \setminus K$. If $a_i \wedge c \neq \mathbf{0}$, then $a_i \wedge b_1 \neq \mathbf{0}$ and $a_i \wedge b_2 \neq \mathbf{0}$ imply $i \in J \cap K$. Thus we get $I_c \subseteqq J \cap K$ and hence $I_c = J \cap K$. Since S is a classification system in L, and since $a_i \leqq b_1 \wedge b_2 = c$ for all $i \in J \cap K$, we can write:

$$\left(\bigvee_{i\in J} a_i\right) \wedge \left(\bigvee_{i\in K} a_i\right) = c = \bigvee_{i\in I}(c\wedge a_i) = \bigvee_{i\in J\cap K}(c\wedge a_i) = \bigvee_{i\in J\cap K} a_i. \qquad \blacksquare$$

A lattice L with $\mathbf{0}$ is called a *pseudocomplemented lattice* if any element $x \in L$ has a pseudocomplement x^*, that is, for any $x \in L$ there exists an element $x^* \in L$ such that $y \wedge x = \mathbf{0} \Leftrightarrow y \leqq x^*$. An element a of a pseudocomplemented lattice L is called a *semicentral element* ([1]) if

(6) $$x = (x \wedge a) \vee (x \wedge a^*) \text{ holds for all } x \in L.$$

Let $\mathrm{Sem}(L)$ stand for the set of semicentral elements of L.

Remark 6.3. Clearly, if L is a bounded lattice, then $\mathbf{0},\mathbf{1}\in\mathrm{Sem}(L)$. In view of [4], if L is a complete pseudocomplemented lattice, then any classification system of L is independent and consist of semicentral elements.

Proposition 6.4. *If L is a CJ-generated complete pseudocomplemented lattice then* $\mathrm{Sem}(L) = \mathcal{B}(L)$.

Proof. It is clear that $\mathbf{0},\mathbf{1}\in \mathcal{B}(L)$. Let $a \in\mathrm{Sem}(L)\setminus\{\mathbf{0},\mathbf{1}\}$. Since, in view of (6), the set $\{a, a^*\}$ is a classification system of L, we have $a \in \mathcal{B}(L)$. Hence, $\mathrm{Sem}(L) \subseteqq \mathcal{B}(L)$. As by Remark 6.3 any box element of L is a semicentral element, we obtain $\mathrm{Sem}(L) = \mathcal{B}(L)$. $\blacksquare$

Theorem 6.5. *Let L be a CJ-generated complete lattice.*

(i) *If the finest classification system of L is independent, then $\mathcal{B}(L)$ is an atomistic complete Boolean sublattice of L.*
(ii) *If $\mathcal{B}(L)$ is a pseudocomplemented lattice, then it is a Boolean sublattice of L and every classification system of L is independent.*

Proof. (i) First we prove that $\mathcal{B}(L)$ is a sublattice of L. As $b_1 \wedge b_2 \in \mathcal{B}(L)$ for any $b_1, b_2 \in \mathcal{B}(L)$, we have to show only $b_1 \sqcup b_2 = b_1 \vee b_2$.

Let $S_0 = \{\alpha_i \mid i \in I_0\}$ be the finest classification system of L. In view of Remark 4.6, there exist $J_1, J_2 \subseteqq I_0$ with $b_1 = \bigvee_{i\in J_1}\alpha_i$ and $b_2 = \bigvee_{i\in J_2}\alpha_i$. Then $b_1 \vee b_2 = \bigvee_{i\in J_1\cup J_2}\alpha_i$. Since S_0 is independent, Corollary 6.2 (i) gives $b_1 \vee b_2 \in \mathcal{B}(L)$. Hence $b_1 \sqcup b_2 = \bigwedge\{b \in \mathcal{B}(L) \mid b_1, b_2 \leqq b\} = b_1 \vee b_2$.

Now let us consider the map $\Phi : \mathcal{P}(I_0) \to \mathcal{B}(L)$, $\Phi(J) = \bigvee_{i\in J}\alpha_i$, $J \subseteqq I_0$. Since Corollary 6.2 (iii) implies $\Phi(J \cap K) = \Phi(J) \wedge \Phi(K)$, Φ is a $\wedge$-homomorphism. As any $b \in \mathcal{B}(L)$ is of the form $b = \bigvee_{i\in M}\alpha_i$ for some $M \subseteqq I_0$ (see Remark 4.6), Φ is onto. We prove that Φ is one-to-one:

Assume that $\Phi(J_1) = \Phi(J_2)$, i.e. $\bigvee_{i\in J_1}\alpha_i = \bigvee_{i\in J_2}\alpha_i$. Then for any $j \in J_1$ we have $\alpha_j \leqq \bigvee_{i\in J_2}\alpha_i$ and Corollary 6.2 (ii) gives $i \in J_2$. Thus $J_1 \subseteqq J_2$. Symmetrically we prove $J_2 \subseteqq J_1$. Hence $J_1 = J_2$.

Thus Φ is a lattice isomorphism and $\mathcal{P}(I_0) \cong \mathcal{B}(L)$. Therefore, $\mathcal{B}(L)$ is an atomistic complete Boolean lattice.

(ii) Let $S = \{a_i \mid i \in I\}$ be an arbitrary classification system of L. Since $S \in \mathrm{Cls}(\mathcal{B}(L))$ and since $\mathcal{B}(L)$ is a pseudocomplemented lattice, S is independent in $\mathcal{B}(L)$. Therefore, we get $a_j \wedge (\bigvee_{i\in I\smallsetminus\{j\}}a_i) \leqq a_j \wedge (\bigsqcup_{i\in I\smallsetminus\{j\}}a_i) = \mathbf{0}$, proving that S is independent in L. Since every classification system of L is independent, S_0 is also independent. Hence, according to the above (i), $\mathcal{B}(L)$ is a Boolean sublattice of L. ■

As a consequence of Theorem 6.5 and Corollary 5.8 we obtain:

Corollary 6.6. *If the finest classification system of a CJ-generated complete lattice L is independent, then every classification system of L is independent and* Cls(L) *is isomorphic to a partition lattice.*

Combining Theorem 6.5, Proposition 6.4 and Corollary 6.6 we obtain:

Corollary 6.7. *Let L be a CJ-generated complete pseudocomplemented lattice. Then the semicentral elements of L form an atomistic complete Boolean sublattice of L which coincides with the box lattice of L. The classification lattice of L is isomorphic to a partition lattice.*

PROBLEMS

Since any partition lattice is a particular geometric lattice, the problems below arise naturally:

1) Is it true that every geometric lattice is isomorphic to a classification lattice ?

2) Characterize those atomistic complete lattices whose classification lattices are geometric lattices (partition lattices).

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