

LEXICO EXTENSION AND A CUT COMPLETION OF A HALF l -GROUP

ŠTEFAN ČERNÁK

Department of Mathematics, Faculty of Civil Engineering
Technical University
Vysokoškolská 4, SK-042 02 Košice, Slovakia
e-mail: svfkm@tuke.sk

AND

MILAN DEMKO

Department of Mathematics, FHPV PU
17 Novembra 1, SK-081 16 Prešov, Slovakia
e-mail: demko@unipo.sk

Abstract

The cut completion of an hl -group G with the abelian increasing part is investigated under the assumption that G is a lexico extension of its hl -subgroup.

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0. INTRODUCTION

The notion of a half l -group as a generalization of the notion of an l -group was introduced and studied by M. Giraudet and F. Lucas [4].

R.N. Ball [1] has defined the notion of a cut completion of an l -group.

In this paper we define the notions of a cut completion and a lexico extension of a half l -group. We prove a theorem on a cut completion of a half l -group having an abelian increasing part which can be expressed as a nontrivial lexico extension. A particular case of this theorem is a result of J. Jakubík [5] dealing with a cut completion of an abelian l -group.

1. PRELIMINARIES

Let G be an abelian l -group. G is called a *lexico extension of its l -subgroup* $A \neq \{0\}$ if

- (i) A is a convex l -subgroup of G ,
- (ii) if $0 < g \in G$, $g \notin A$, then $g > a$ for each $a \in A$.

If G is a lexico extension of A , we shall write $G = \langle A \rangle$. If $G = \langle A \rangle$, then A is an l -ideal of G and (cf. [3] and [2])

- (a) A is comparable to all convex l -subgroups of G (i.e., if A' is a convex l -subgroup of G then either $A' \subseteq A$ or $A \subseteq A'$).
- (b) G/A is a linearly ordered group.

Let G be a group and a partially ordered set. Set

$$G \uparrow = \{g \in G : x \leq y \Rightarrow g + x \leq g + y \text{ for all } x, y \in G\},$$

$$G \downarrow = \{g \in G : x \leq y \Rightarrow g + x \geq g + y \text{ for all } x, y \in G\}.$$

$G \uparrow (G \downarrow)$ is called the *increasing (decreasing)* part of G .

G is said to be a *half l -group* (abbreviated to an *hl-group*) if the following conditions are satisfied (cf. [4]):

- (i) the partial order \leq on G is non-trivial,
- (ii) if $x, y, g \in G$ and $x \leq y$, then $x + g \leq y + g$,
- (iii) $G = G \uparrow \cup G \downarrow$,
- (iv) $G \uparrow$ is an l -group.

If $G \uparrow$ is a linearly ordered group, then hl -group G will be called a *half linearly ordered group*.

Every l -group $G \neq \{0\}$ is a special case of an hl -group with $G \downarrow = \emptyset$.

We denote by \mathcal{HL} the class of all hl -groups that fail to be l -groups.

The following results will be applied in the next.

Proposition 1.1 (cf. [4]). *Let $G \in \mathcal{HL}$. Then*

- (i) $G \uparrow$ is a subgroup of the group G and $G \uparrow$ has index 2,
- (ii) $G \uparrow$ and $G \downarrow$ are isomorphic groups and also dually isomorphic lattices,
- (iii) if $x \in G \uparrow$ and $y \in G \downarrow$, then x and y are incomparable,
- (iv) the set $\{g \in G : g \neq 0, 2g = 0\}$ is nonempty.

Let G be an hl -group. A subgroup $A \neq \{0\}$ of G is called a half l -subgroup (abbreviated to an hl -subgroup) if $A \uparrow = A \cap G \uparrow$ is an l -subgroup of $G \uparrow$. If A is an hl -subgroup (proper hl -subgroup) of G we use the notation $A \leq G$ ($A < G$). We say that an hl -subgroup A of G is convex in G if $A \uparrow$ is convex in $G \uparrow$. A convex hl -subgroup A of G is said to be an hl -ideal of G if $A \uparrow$ is a normal subgroup of G . According to 1.1 $G \uparrow$ is an hl -ideal of G .

Let G be an hl -group, $G \in \mathcal{HL}$ and $A \uparrow$ an hl -ideal of G , $A \in \mathcal{HL}$. We can form the factor group $\overline{G} = G/A \uparrow$. For elements $g_1 + A \uparrow, g_2 + A \uparrow \in \overline{G}$, we put $g_1 + A \uparrow \leq g_2 + A \uparrow$ if and only if there exist $g'_1 \in g_1 + A \uparrow$ and $g'_2 \in g_2 + A \uparrow$ with $g'_1 \leq g'_2$. Then \overline{G} is a partially ordered set and to each $g'_1 \in g_1 + A \uparrow$ there exists $g'_2 \in g_2 + A \uparrow$ such that $g'_1 \leq g'_2$. It can be easily verified that if $A < G$, then \overline{G} is an hl -group with the increasing part $\overline{G} \uparrow = \{g + A \uparrow : g \in G \uparrow\}$ and decreasing part $\overline{G} \downarrow = \{g + A \uparrow : g \in G \downarrow\}$.

If $A = G$ then \overline{G} is trivially ordered. Hence \overline{G} fails to be an hl -group.

A 1-1 mapping φ from an hl -group G onto an hl -group G' is called an hl -isomorphism if φ is a group homomorphism and if $\varphi|G \uparrow$ is a lattice homomorphism of $G \uparrow$ onto $G' \uparrow$.

2. LEXICO EXTENSION OF AN hl -SUBGROUP

Let G be an hl -group, $G \in \mathcal{HL}$ with the abelian increasing part $G \uparrow$. Let A be an hl -subgroup of G , $A \in \mathcal{HL}$. If $G \uparrow$ is a lexico extension of $A \uparrow$, then we say that G is a lexico extension of A and we express this situation by writing $G = \langle A \rangle_h$.

Lemma 2.1. *Let $G = \langle A \rangle_h$. Then*

- (i) A is an hl -ideal of G ,
- (ii) if $A < G$, then $\overline{G} = G/A \uparrow$ is a half linearly ordered group.

Proof. (i) We have to show that $A \uparrow$ is normal in G . Since $A \uparrow$ is a convex l -subgroup of $G \uparrow$, $-g + A \uparrow + g$ ($g \in G$) is a convex subset of $G \uparrow$. It is a routine to verify that $-g + A \uparrow + g$ is a subgroup of $G \uparrow$. Let $-g + a_1 + g$, $-g + a_2 + g \in -g + A \uparrow + g$, $g \in G$, and $a_1, a_2 \in A \uparrow$. It is easy to verify that in $G \uparrow$ we have $(-g + a_1 + g) \vee (-g + a_2 + g) = -g + (a_1 \vee a_2) + g$ for each $g \in G \uparrow$, $(-g + a_1 + g) \wedge (-g + a_2 + g) = -g + (a_1 \wedge a_2) + g$ for each $g \in G \downarrow$ and dually. Hence $-g + A \uparrow + g$ is a sublattice of $G \uparrow$ for each $g \in G$. By summarizing we have that $-g + A \uparrow + g$ is a convex l -subgroup of $G \uparrow$ for each $g \in G$. By (a), $A \uparrow$ and $-g + A \uparrow + g$ are comparable. The fact that $G \uparrow$ is abelian implies $-g + A \uparrow + g = A \uparrow$ for all $g \in G \uparrow$. Suppose that $g \in G \downarrow$ and $A \uparrow \subseteq -g + A \uparrow + g$. Let $a \in A \uparrow$. Then $a = -g + a_0 + g$, where $a_0 \in A \uparrow$ and hence $-g + a + g = -2g + a_0 + 2g$. Since $2g \in G \uparrow$, we get $-g + a + g \in A \uparrow$. Thus $-g + A \uparrow + g \subseteq A \uparrow$ for all $g \in G$. Therefore, $A \uparrow$ is normal in G .

(ii) follows from the property (b) of a lexico extension. \blacksquare

If $G = \langle A \rangle_h$, then Lemma 2.1 yields that $A \uparrow$ is a normal subgroup of G , but A need not be normal in G .

Examples. Let M be the set of all functions $f : R \rightarrow R$; $f(x) = \pm x + k$, $k \in R$. If a binary operation on M is defined as a composition (i.e., $fg(x) = f(g(x))$ for all $x \in R$) and a binary relation \leq on M is defined pointwise, then M is a half linearly ordered group, with $M \uparrow = \{f : f(x) = x + k\}$ and $M \downarrow = \{f : f(x) = -x + k\}$. Now, let $H = \{(f_1, f_2) : f_1, f_2 \in M \uparrow\}$. For each $(f_1, f_2), (f'_1, f'_2) \in H$ we put $(f_1, f_2) \leq (f'_1, f'_2)$ if and only if either $f_1 < f'_1$ or $f_1 = f'_1$ and $f_2 \leq f'_2$. Then H is a linearly ordered set that is called the *lexicographic product* of the two linearly ordered sets $M \uparrow$ and we use the denotation $H = M \uparrow \circ M \uparrow$. Analogously we can construct $K = M \downarrow \circ M \downarrow$. If a binary operation on H is defined componentwise, then H is a linearly ordered group. Therefore, $G = H \cup K$ is a half linearly ordered group with $G \uparrow = H, G \downarrow = K$. Let $A_1 = \{(id, g) \in G : g \in M \uparrow\}$ (id is an identity function) and $A_2 = \{(g_1, g) : g_1 \text{ is a fixed element of } M \downarrow, g \in M \downarrow\}$. Then $A = A_1 \cup A_2$ is a half linearly ordered group, $A \uparrow = A_1, A \downarrow = A_2$. We have $G = \langle A \rangle_h$, but A fails to be normal in G . In fact, for all $f \in M, f \neq id, g_1$ we have $f^{-1}g_1f \neq g_1$, thus $(f, f)^{-1}(g_1, g)(f, f) \notin A_2$ for each $(g_1, g) \in A_2$.

Assume that G is an hl -group, $G \in \mathcal{HL}$ and that $A \in \mathcal{HL}$ is an hl -ideal of G such that A is a normal subgroup of G . Define a partial order on the factor group G/A (and also on the factor group $G \uparrow / A \uparrow$) analogously

as above on $G/A \uparrow$. Then G/A is a lattice ordered group. The mapping $f : G \uparrow / A \uparrow \rightarrow G/A$ defined by $f(g+A \uparrow) = g+A, g \in G \uparrow$ is an isomorphism of the lattice ordered group $G \uparrow / A \uparrow$ onto G/A .

Suppose that $G = \langle A \rangle_h$ and that A is normal in G . Then, by the property (b) of a lexico extension, we have that G/A is a linearly ordered group.

Lemma 2.2. *Let $G = \langle A \rangle_h, g_1+A \uparrow, g_2+A \uparrow \in \overline{G}$, and let $g_1+A \uparrow < g_2+A \uparrow$. Then $g_1 < g_2$.*

Proof. From $g_1 + A \uparrow < g_2 + A \uparrow$ it follows that either $g_1, g_2 \in G \uparrow$ or $g_1, g_2 \in G \downarrow$. Now, let $g_1, g_2 \in G \uparrow$. There exists $g'_2 \in g_2 + A \uparrow$ such that $g_1 < g'_2$. Hence $g'_2 \in G \uparrow$ and $g'_2 - g_1 > 0$. Since $g'_2 - g_1 \notin A \uparrow$ and $g'_2 - g_2 \in A \uparrow$, we get $g'_2 - g_1 > g'_2 - g_2$. Hence $-g_1 > -g_2$ and $g_1 < g_2$. Suppose that $g_1, g_2 \in G \downarrow$. There exists $g''_2 \in g_2 + A \uparrow$ with $g_1 < g''_2$. Hence $g''_2 \in G \downarrow, g''_2 - g_1 \in G \uparrow, g''_2 - g_1 \notin A \uparrow$ and $g''_2 - g_1 > 0$. Further, we have $g''_2 - g_2 \in A \uparrow$. Then $g''_2 - g_1 > g''_2 - g_2$ implies that $-g_1 < -g_2$ and so $g_1 < g_2$. ■

Corollary. *Let $G = \langle A \rangle_h$. Then G is a half linearly ordered group if and only if A is a half linearly ordered group.*

For the remaining part of this section, we assume that G, A and B are hl -groups from \mathcal{HL} such that

- (I) $G \uparrow$ and $B \uparrow$ are abelian l -groups,
- (II) $G = \langle A \rangle_h, A < G$,
- (III) A is an hl -subgroup of B ,
- (IV) $G \cap B = A$.

According to Proposition 1.1, there exists an element $a \in A \downarrow$ of order 2.

Form the set

$$H_0 = \{(g, b) : \text{either } g \in G \uparrow, b \in B \uparrow \text{ or } g \in G \downarrow, b \in B \downarrow\}.$$

For elements $(g_1, b_1), (g_2, b_2) \in H_0$, we set

$$(g_1, b_1) \equiv (g_2, b_2)$$

if $g_1 - g_2 \in A \uparrow, b_1 - b_2 \in A \uparrow, g_1 - g_2 = b_2 - b_1$ and if either $g_1, g_2 \in G \uparrow, b_1, b_2 \in B \uparrow$ or $g_1, g_2 \in G \downarrow, b_1, b_2 \in B \downarrow$.

The relation \equiv is an equivalence. It is clear that the relation \equiv is reflexive and symmetric. To establish the transitivity, suppose that $(g_1, b_1) \equiv (g_2, b_2)$, $(g_2, b_2) \equiv (g_3, b_3)$. We will consider only the following case. Let $g_1, g_2 \in G \downarrow$, and $b_1, b_2 \in B \downarrow$. Then $g_3 \in G \downarrow$, and $b_3 \in B \downarrow$. We have $g_1 - g_2 \in A \uparrow$, $b_2 - b_1 \in A \uparrow$, $g_1 - g_2 = b_2 - b_1$, $g_2 - g_3 \in A \uparrow$, $b_3 - b_2 \in A \uparrow$, and $g_2 - g_3 = b_3 - b_2$. By (I) and (II), $A \uparrow$ is abelian. Then $g_1 - g_3 = (g_1 - g_2) + (g_2 - g_3) = (b_2 - b_1) + (b_3 - b_2) = (b_3 - b_2) + (b_2 - b_1) = b_3 - b_1$. Hence $g_1 - g_3 \in A \uparrow$ and $b_3 - b_1 \in A \uparrow$. Therefore $(g_1, b_1) \equiv (g_3, b_3)$.

Denote

$$\begin{aligned} \overline{(g, b)} &= \{(g', b') \in H_0 : (g, b) \equiv (g', b')\}, \\ H &= \{\overline{(g, b)} : (g, b) \in H_0\}. \end{aligned}$$

Let $\overline{(g_1, b_1)}, \overline{(g_2, b_2)} \in H$. We put

$$\overline{(g_1, b_1)} + \overline{(g_2, b_2)} = \overline{(g_1 + g_2, b_1 + b_2)}.$$

The binary operation $+$ on H is correctly defined; $\overline{(0, 0)}$ is a neutral element and $\overline{(-g, -b)}$ is an inverse to $\overline{(g, b)}$.

We have

Lemma 2.3. $(H, +)$ is a group. ■

Let $\overline{(g_1, b_1)}, \overline{(g_2, b_2)} \in H$. We put

$$\overline{(g_1, b_1)} \leq \overline{(g_2, b_2)}$$

if either $g_1 < g_2$ and $g_1 - g_2 \notin A \uparrow$ or $g_1 - g_2 \in A \uparrow$ and $g_1 - g_2 \leq b_2 - b_1$.

The definition implies that either $g_1, g_2 \in G \uparrow$ or $g_1, g_2 \in G \downarrow$. Now we verify that the relation \leq is correctly defined. Let $\overline{(g'_1, b'_1)} = \overline{(g_1, b_1)}$, $\overline{(g'_2, b'_2)} = \overline{(g_2, b_2)}$.

Assume that $g_1 < g_2$, $g_1 - g_2 \notin A \uparrow$. Then $g_1 + A \uparrow < g_2 + A \uparrow$ and $g_1, g_2 \in G \uparrow$ or $g_1, g_2 \in G \downarrow$. Since $g_1 - g'_1 \in A \uparrow$ and $g_2 - g'_2 \in A \uparrow$, we get $g_1 + A \uparrow = g'_1 + A \uparrow$ and $g_2 + A \uparrow = g'_2 + A \uparrow$. With respect to Lemma 2.2, we get $g'_1 < g'_2$. Suppose that $g'_1 - g'_2 \in A \uparrow$. Then $g_1 - g_2 = (g_1 - g'_1) + (g'_1 - g'_2) + (g'_2 - g_2) \in A \uparrow$, a contradiction. Hence $g'_1 - g'_2 \notin A \uparrow$.

Assume that $g_1 - g_2 \in A \uparrow$, $g_1 - g_2 \leq b_2 - b_1$. Then $g'_1 - g'_2 = (g'_1 - g_1) + (g_1 - g_2) + (g_2 - g'_2) \leq (b_1 - b'_1) + (b_2 - b_1) + (b'_2 - b_2) = (b_1 - b'_1) + (b'_2 - b_2) + (b_2 - b_1) = (b_1 - b'_1) + (b'_2 - b_1) = (b'_2 - b_1) + (b_1 - b'_1) = b'_2 - b'_1$. We also have shown that $g'_1 - g'_2 \in A \uparrow$.

It is evident that the relation \leq is reflexive.

Let $\overline{(g_1, b_1)} \leq \overline{(g_2, b_2)}$, $\overline{(g_2, b_2)} \leq \overline{(g_1, b_1)}$. Then $g_1 - g_2 \in A \uparrow$, $g_1 - g_2 \leq b_2 - b_1$ and $g_2 - g_1 \leq b_1 - b_2$. Hence $g_1 - g_2 = b_2 - b_1$ and so $\overline{(g_1, b_1)} = \overline{(g_2, b_2)}$.

The antisymmetry is satisfied.

Let $\overline{(g_1, b_1)} \leq \overline{(g_2, b_2)}$, $\overline{(g_2, b_2)} \leq \overline{(g_3, b_3)}$.

(α) Assume that $g_1 < g_2$, $g_1 - g_2 \notin A \uparrow$, $g_2 < g_3$, and $g_2 - g_3 \notin A \uparrow$. We will consider only the case that $g_1, g_2 \in G \downarrow$. Then also $g_3 \in G \downarrow$ and $g_1 < g_3$. Assume that $g_1 - g_3 \in A \uparrow$. Then $g_1 + A \uparrow = g_3 + A \uparrow$. Since $g_1 + A \uparrow$ is a convex subset of $G \downarrow$ and $g_1 < g_2 < g_3$, we obtain $g_2 \in g_1 + A \uparrow$. Hence $g_1 - g_2 \in A \uparrow$, a contradiction.

(β) Assume that $g_1 - g_2 \in A \uparrow$, $g_1 - g_2 \leq b_2 - b_1$, $g_2 - g_3 \in A \uparrow$, and $g_2 - g_3 \leq b_3 - b_2$. Then $g_1 - g_3 = (g_1 - g_2) + (g_2 - g_3) \in A \uparrow$, and $g_1 - g_3 = (g_1 - g_2) + (g_2 - g_3) \leq (b_2 - b_1) + (b_3 - b_2) = (b_3 - b_2) + (b_2 - b_1) = b_3 - b_1$.

(γ) Assume that $g_1 < g_2$, $g_1 - g_2 \notin A \uparrow$, $g_2 - g_3 \in A \uparrow$, and $g_2 - g_3 \leq b_3 - b_2$. We will consider only the case $g_1, g_2 \in G \downarrow$. Hence $g_3 \in G \downarrow$, $g_2 - g_1 > 0$ and $g_2 - g_1 \notin A \uparrow$. From this it follows that $g_2 - g_1 > g_2 - g_3$, $-g_1 < -g_3$ and $g_1 < g_3$. Suppose that $g_1 - g_3 \in A \uparrow$. Then $g_1 - g_2 = (g_1 - g_3) + (g_3 - g_2) \in A \uparrow$, a contradiction.

(δ) Suppose that $g_1 - g_2 \in A \uparrow$, $g_1 - g_2 \leq b_2 - b_1$, $g_2 < g_3$, and $g_2 - g_3 \notin A \uparrow$. The case is analogous to (γ).

In all cases (α)-(δ) we get $\overline{(g_1, b_1)} \leq \overline{(g_3, b_3)}$, i.e the relation \leq is transitive.

We have shown that the following lemma is valid.

Lemma 2.4. (H, \leq) is a partially ordered set. ■

Lemma 2.5. Let $\overline{(g_1, b_1)}, \overline{(g_2, b_2)}, \overline{(g_3, b_3)} \in H$, $\overline{(g_1, b_1)} \leq \overline{(g_2, b_2)}$. Then $\overline{(g_1, b_1)} + \overline{(g_3, b_3)} \leq \overline{(g_2, b_2)} + \overline{(g_3, b_3)}$.

Proof. We will consider only the case that $g_1, g_2, g_3 \in G \downarrow$.

Suppose that $g_1 < g_2$ and $g_1 - g_2 \notin A \uparrow$. Then $g_1 + g_3 < g_2 + g_3$ and $(g_1 + g_3) - (g_2 + g_3) = g_1 - g_2 \notin A \uparrow$.

Assume that $g_1 - g_2 \in A \uparrow$ and $g_1 - g_2 \leq b_2 - b_1$. Then $(g_1 + g_3) - (g_2 + g_3) \in A \uparrow$ and $(g_1 + g_3) - (g_2 + g_3) = g_1 - g_2 \leq b_2 - b_1 = (b_2 + b_3) - (b_1 + b_3)$. Therefore $\overline{(g_1, b_1)} + \overline{(g_3, b_3)} \leq \overline{(g_2, b_2)} + \overline{(g_3, b_3)}$. ■

Form the sets

$$H \uparrow = \{\overline{(g, b)} : g \in G \uparrow, b \in B \uparrow\}, \quad H \downarrow = \{\overline{(g, b)} : g \in G \downarrow, b \in B \downarrow\}.$$

Then we have

Lemma 2.6. $H = (H \uparrow) \cup (H \downarrow)$. ■

Lemma 2.7. $H \uparrow$ is an increasing part and $H \downarrow$ is a decreasing part of H .

Proof. Assume that $\overline{(g_1, b_1)}, \overline{(g_2, b_2)} \in H$, $\overline{(g_1, b_1)} \leq \overline{(g_2, b_2)}$ and $\overline{(g_3, b_3)} \in H \downarrow$. We intend to show that $H \downarrow$ is a decreasing part of H , i.e., that $\overline{(g_3, b_3)} + \overline{(g_2, b_2)} \leq \overline{(g_3, b_3)} + \overline{(g_1, b_1)}$ is valid.

Let $g_1 < g_2$, $g_1 - g_2 \notin A \uparrow$. Then $g_3 + g_1 > g_3 + g_2$. Suppose that $(g_3 + g_2) - (g_3 + g_1) \in A \uparrow$. With respect to Lemma 2.1, $A \uparrow$ is normal in G . Thus $g_2 - g_1 \in -g_3 + A \uparrow + g_3 \subseteq A \uparrow$. Hence $g_1 - g_2 \in A \uparrow$, a contradiction.

Let $g_1 - g_2 \in A \uparrow$ and $g_1 - g_2 \leq b_2 - b_1$. By using the normality of $A \uparrow$ in G , we obtain $g_3 + g_2 - (g_3 + g_1) = g_3 + (g_2 - g_1) - g_3 \in A \uparrow$. There exist elements $g'_3 \in G \uparrow$ and $b'_3 \in B \uparrow$ such that $g_3 = a + g'_3$, $b_3 = a + b'_3$. From $g_2 - g_1 \geq b_1 - b_2$, it follows $a + g'_3 - g'_3 + g_2 - g_1 + a \leq a + b'_3 - b'_3 + b_1 - b_2 + a$, $(a + g'_3 + g_2) - (a + g'_3 + g_1) \leq (a + b'_3 + b_1) - (a + b'_3 + b_2)$ and $(g_3 + g_2) - (g_3 + g_1) \leq (b_3 + b_1) - (b_3 + b_2)$.

In an analogous way, we show that $H \uparrow$ is an increasing part of H . ■

$H \uparrow$ is a group (subgroup of H) and a partially ordered set (a partial order is inherited from H). Then according to Lemmas 2.5 and 2.7, $H \uparrow$ is a partially ordered group.

Lemma 2.8. $H \uparrow$ is an l -group.

Proof. It is sufficient to prove that there exists $\sup\{\overline{(0, 0)}, \overline{(g, b)}\}$ for each $\overline{(g, b)} \in H \uparrow$. If $g \notin A \uparrow$ then $g > 0$ or $g < 0$. Hence $\overline{(g, b)}$ and $\overline{(0, 0)}$ are comparable. If $g \in A \uparrow$ then $g + b \in B \uparrow$ and $\overline{(g, b)} = \overline{(0, g + b)}$. Let

$b' = \sup\{0, g + b\}$ in $B \uparrow$. By using the same procedure as in the proof of Lemma 2.4 in [5], we obtain $\overline{(0, b')} = \sup\{\overline{(0, 0)}, \overline{(g, b)}\}$. ■

From Lemmas 2.3–2.8 it follows

Lemma 2.9. *H is an hl -group, $H \in \mathcal{HL}$.* ■

Recall that there is $a \in A \downarrow$, an element of order 2 (by Proposition 1.1), and that, by (IV), $A \subseteq B$. Define the mapping φ of G into H by $\varphi(g) = \overline{(g, 0)}$ if $g \in G \uparrow$ and $\varphi(g) = \overline{(g, a)}$ if $g \in G \downarrow$. Then φ is an hl -isomorphism of the hl -group G into H .

If we put $\psi(b) = \overline{(0, b)}$ for each $b \in B \uparrow$ and $\psi(b) = \overline{(a, b)}$ for each $b \in B \downarrow$, then ψ is an hl -isomorphism of the hl -group B into H .

If $x \in G \cap B$ then $\varphi(x) = \psi(x)$. In fact, if $x \in (G \cap B) \uparrow = (G \uparrow) \cap (B \uparrow)$, then $\varphi(x) = \overline{(x, 0)} = \overline{(0, x)} = \psi(x)$; and if $x \in (G \cap B) \downarrow = (G \downarrow) \cap (B \downarrow)$, then $\varphi(x) = \overline{(x, a)} = \overline{(a, x)} = \psi(x)$.

In the next, we shall identify elements g and $\varphi(g)$ for each $g \in G$ and also b and $\psi(b)$ for each $b \in B$. Then G and B are hl -subgroups of H .

Lemma 2.10. $H = \langle B \rangle_h$.

Proof. B is an hl -subgroup of H . We have to prove that $H \uparrow = \langle B \uparrow \rangle$. Assume that $\overline{(g, b)} \in H \uparrow$, $\overline{(0, b')} \in B \uparrow$, $\overline{(0, 0)} \leq \overline{(g, b)} \leq \overline{(0, b')}$. Then $g \in A \uparrow \subseteq B \uparrow$ and so $g + b \in B \uparrow$, $\overline{(g, b)} = \overline{(0, g + b)} \in B \uparrow$. Hence $B \uparrow$ is a convex l -subgroup of $H \uparrow$. Let $\overline{(0, 0)} < \overline{(g, b)} \in H \uparrow$, $\overline{(g, b)} \notin B \uparrow$. Then $g \notin A \uparrow$. Therefore, $g > 0$ and thus $\overline{(0, b')} < \overline{(g, b)}$ for each $\overline{(0, b')} \in B \uparrow$. ■

By using Lemmas 2.10 and 2.1, B is an hl -ideal of H . Therefore, we can form the factor hl -group $\overline{H} = H/B \uparrow$.

Lemma 2.11. *Half l -groups \overline{G} and \overline{H} are hl -isomorphic.*

Proof. Define the mapping $f : \overline{G} \rightarrow \overline{H}$ by $f(g + A \uparrow) = g + B \uparrow$. Let $g + A \uparrow = g' + A \uparrow$. Then $g - g' \in A \uparrow \subset B \uparrow$. Thus $g + B \uparrow = g' + B \uparrow$. Therefore the mapping f is correctly defined.

Let $g_1 + A \uparrow, g_2 + A \uparrow \in \overline{G}$. Then $f((g_1 + A \uparrow) + (g_2 + A \uparrow)) = f((g_1 + g_2) + A \uparrow) = (g_1 + g_2) + B \uparrow = (g_1 + B \uparrow) + (g_2 + B \uparrow) = f(g_1 + A \uparrow) + f(g_2 + A \uparrow)$.

Assume that $g_1, g_2 \in G$ and $f(g_1 + A \uparrow) = f(g_2 + A \uparrow)$. From $g_1 + B \uparrow = g_2 + B \uparrow$, we infer that either $g_1, g_2 \in G \uparrow$ or $g_1, g_2 \in G \downarrow$. Hence $g_1 - g_2 \in G \uparrow \cap B \uparrow = A \uparrow$ and so $g_1 + A \uparrow = g_2 + A \uparrow$.

Let $(g, b) + B \uparrow \in \overline{H}$. Assume that $(g, b) \in H \downarrow$. Hence $g \in G \downarrow$. Recall that g is identified with (g, a) . As for $(g, a) - (g, b) = (0, a - b) \in B \uparrow$, we have $(g, a) + B \uparrow = (g, b) + B \uparrow$. Therefore $f(g + A \uparrow) = (g, b) + B \uparrow$. If $(g, b) \in H \uparrow$, the proof is similar.

We have shown that f is a group isomorphism of \overline{G} onto \overline{H} .

Assume that $g_1 + A \uparrow, g_2 + A \uparrow \in \overline{G}$ and $g_1 + A \uparrow \leq g_2 + A \uparrow$. If $g_1 + A \uparrow = g_2 + A \uparrow$, then $f(g_1 + A \uparrow) = f(g_2 + A \uparrow)$. Let $g_1 + A \uparrow < g_2 + A \uparrow$. By Lemma 2.2, $g_1 < g_2$. Hence $f(g_1 + A \uparrow) = g_1 + B \uparrow < g_2 + B \uparrow = f(g_2 + A \uparrow)$. The converse is similar. ■

Summarizing the previous results, we have

Theorem 2.12. *Let A, B and G be hl-groups from \mathcal{HL} satisfying (I)–(IV). Then there exists an hl-group $H \in \mathcal{HL}$ such that*

- (i) $H = \langle B \rangle_h$,
- (ii) G is an hl-subgroup of H ,
- (iii) hl-groups $G/A \uparrow$ and $H/B \uparrow$ are hl-isomorphic.

3. CUT COMPLETION OF A LEXICO EXTENSION

Let G be an hl-group. A subset X of $G \uparrow$ is said to be a *cut of $G \uparrow$* if X is an order closed (i.e., $g = \bigvee S$ for $S \subseteq X$ implies $g \in X$) lattice ideal of G such that $g + X \neq X \neq X + g$ for any $0 < g \in G$. A *cut of $G \downarrow$* is defined in the same way. If X is a cut either of $G \uparrow$ or of $G \downarrow$, then X is called a cut of G .

$G \uparrow (G \downarrow)$ is said to be *cut complete* if every cut of $G \uparrow (G \downarrow)$ has a supremum in $G \uparrow (G \downarrow)$.

Remark that if $Z \subseteq G \uparrow (Z \subseteq G \downarrow)$, then $\sup(Z)$ exists in $G \uparrow (G \downarrow)$ if and only if $\sup(Z)$ exists in G , and $\sup(Z)$ in G is equal to $\sup(Z)$ in $G \uparrow (G \downarrow)$.

G is called *cut complete* provided that every cut of G has a supremum in G .

An *hl*-subgroup G' of G will be said to be order dense in G if for every element $0 < g \in G$ there exists $g' \in G'$ with $0 < g' \leq g$.

An *hl*-group G^C is said to be a *cut completion* of G if the following conditions are satisfied:

- (i) G^C is cut complete;
- (ii) G is an order dense *hl*-subgroup of G^C ;
- (iii) if K is an *hl*-subgroup of G^C such that $G \leq K < G^C$, then K is not cut complete.

Since $G \uparrow$ and $G \downarrow$ are dually isomorphic lattices, we have

Lemma 3.1. *For any *hl*-group G , $G \in \mathcal{HL}$ is cut complete if and only if $G \uparrow$ is cut complete.* ■

Lemma 3.2 ([5], Lemma 3.1). *Let G be an abelian *l*-group, $G = \langle A \rangle$, $A \neq \{0\}$. If A is cut complete, then G is cut complete.* ■

Lemma 3.3. *Let H and B be *hl*-groups from \mathcal{HL} such that $H \uparrow$ is abelian and $H = \langle B \rangle_h$. If B is cut complete, then H is cut complete.*

Proof. The assumption that B is cut complete and Lemma 3.1 yield that $B \uparrow$ is cut complete. From $H \uparrow = \langle B \uparrow \rangle$, $B \uparrow \neq \{0\}$ and Lemma 3.2, we obtain that $H \uparrow$ is cut complete. Hence H is cut complete. ■

Lemma 3.4. *Let A, B, G satisfy the conditions (I)–(IV) and H be such as in Theorem 2.12. Suppose that $B = A^C$. Then $H = G^C$.*

Proof. In view of Lemma 2.10, we have $H = \langle B \rangle_h$. A is order dense in B and B is order dense in H . This yields that A is order dense in H . Thus G is order dense in H . From Lemma 3.3, it follows that H is cut complete. Assume that K is an *hl*-subgroup of H such that $G \leq K < H$. Then $K \uparrow$ is an *l*-subgroup of $H \uparrow$ with $G \uparrow \leq K \uparrow < H \uparrow$. In the same way as in the proof of Lemma 3.2 in [5], it can be shown that $K \uparrow$ fails to be cut complete. Then, by Lemma 3.1, K is not cut complete. Therefore $H = G^C$. ■

Theorem 3.5. *Let $G = \langle A \rangle_h$, and $A < G$. Then*

- (i) $G^C = \langle A^C \rangle_h$,
- (ii) *hl*-groups $G/A \uparrow$ and $G^C/A^C \uparrow$ are *hl*-isomorphic.

Proof. (i): Put $B = A^C$. Let H be as in Theorem 2.12. Then $H = \langle B \rangle_h$ holds. Applying Lemma 3.4, we get that $G^C = \langle A^C \rangle_h$ is valid.

(ii): Immediately follows from Theorem 2.12. ■

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