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ON GENERALIZED Hom-FUNCTORS OF CERTAIN SYMMETRIC MONOIDAL CATEGORIES

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In memory of

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Abstract

It is well-known that for each object A of any category $\mathcal{C}$ there is the covariant functor $H^A : \mathcal{C} \to Set$, where $H^A(X)$ is the set $\mathcal{C}[A, X]$ of all morphisms out of A into X in $\mathcal{C}$ for an arbitrary object $X \in |\mathcal{C}|$ and $H^A(\varphi)$, $\varphi \in \mathcal{C}[X, Y]$, is the total function from $\mathcal{C}[A, X]$ into $\mathcal{C}[A, Y]$ defined by $\mathcal{C}[A, X] \ni u \mapsto u\varphi \in \mathcal{C}[A, Y]$.

If $\underline{\mathcal{C}}$ is a dts-category, then H^A is in a natural manner a d-monoidal functor with respect to

$$\widetilde{H^A} = \Big(\widetilde{H^A}\langle X, Y\rangle : \mathcal{C}[A, X] \times \mathcal{C}[A, Y] \to \mathcal{C}[A, X \otimes Y],$$
$$((u_1, u_2) \mapsto d_A(u_1 \otimes u_2)\Big) \mid X, Y \in |\mathcal{C}|)$$

and

$i_{H^A} : \{\emptyset\} \to \mathcal{C}[A, I]$, $(\emptyset \mapsto t_A)$.

This construction can be generalized to functors H^e from any $dhth\nabla s$-category $\underline{K}$ into the category $\underline{Par}$ related to arbitrary subidentities e of $\underline{K}$ (cf. Schreckenberger [3]). Each such generalized Hom-functor H^e related to any subidentity $e \leq 1_A$, $o_{A,A} \neq e$, turns out to be a monoidal $dhth\nabla s$-functor from $\underline{K}$ into $\underline{Par}$.

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1. Introduction

The development of a functorial semantic of partial algebras requires the knowledge about functors between certain symmetric monoidal categories which preserve the special monoidal structure except for isomorphisms. In [8] was shown that each functor between diagonal-halfterminal-halfdiagonal-inversional-symmetric monoidal categories ($dhth\nabla s$-categories), which respects the monoidal structure and the diagonal morphisms with regard to a morphism family $\widetilde{F}$ of the image category, also preserves the canonical partial order relation, the totality and the injectivity of morphisms, and the terminal morphisms as well as the diagonal inversion morphisms with respect to the same family of isomorphisms $\widetilde{F}$.

The morphism class of a category K will be denoted by K too, the object class of K by $|K|$, and the set of all morphisms in K between objects A and B by $K[A, B]$.

Definition 1.1. Let $K^\bullet$ be a symmetric monoidal category in the sense of Eilenberg-Kelly [1].

A sequence $(K^\bullet; d)$ is called *diagonal-symmetric monoidal category* (shortly *ds-category*; see [6]), if $d = (d_A \in K[A, A \otimes A] \mid A \in |K|)$ is a family of morphisms of K such that

(D1) $\forall A, A' \in |K|\ \forall \varphi \in K[A, A']\ (\varphi d_{A'} = d_A(\varphi \otimes \varphi))$,

(D2) $\forall A \in |K|\ (d_A(d_A \otimes 1_A) = d_A(1_A \otimes d_A)a_{A,A,A})$,

(D3) $\forall A \in |K|\ (d_A s_{A,A} = d_A)$,

(D4) $\forall A, B \in |K|\ ((d_A \otimes d_B)b_{A,A,B,B} = d_{A\otimes B})$

are fulfilled.

$(K^\bullet, d, t)$ is called *diagonal-terminal-symmetric monoidal category* (*dts-category*; see [6]), if $(K^\bullet, d)$ is a ds-category with a family $t = (t_A \mid A \in |K|)$ of terminal morphisms $t_A \in K[A, I]$ such that the conditions

(T1) $\forall A, A' \in |K|\ \forall \varphi \in K[A, A']\ (\varphi t_{A'} = t_A)$ and

(DTR) $\forall A \in |K|\ (d_A(1_A \otimes t_A)r_A = 1_A)$

are right.

$(K^\bullet; d, t, o)$ will be called *diagonal-halfterminal-symmetric monoidal category* (shortly *dhts-category*; see [2], [4], [6]), if d is a morphism family as above, $t = (t_A \in K[A, I] \mid A \in |K|)$ is a family of morphisms in K, and $o : I \to O$ is a distinguished morphism in K related to a distiguished object $O \in |K|, O \neq I$, such that

(D1) $\forall A, A' \in |K|\ \forall \varphi \in K[A, A']\ (d_A(\varphi \otimes \varphi) = \varphi d_{A'})$,

(DTR) $\forall A \in |K|\ (d_A(1_A \otimes t_A)r_A = 1_A)$,

(DTL) $\forall A \in |K|\ (d_A(t_A \otimes 1_A)l_A = 1_A)$,

(DTRL) $\forall A_1, A_2 \in |K|\ (d_{A_1 \otimes A_2}((1_{A_1} \otimes t_{A_2})r_{A_1} \otimes (t_{A_1} \otimes 1_{A_2})l_{A_2}) = 1_{A_1 \otimes A_2}))$,

(TT) $\forall A, B \in |K|\ (t_{A \otimes B} = (t_A \otimes t_B)t_{I \otimes I})$,

(O1) $\forall A \in |K|\ (A \otimes O = O \otimes A = O)$,

(o1) $\forall A \in |K|\ \forall \varphi \in K[A, O]\ (t_A o = \varphi)$, and

(o2) $\forall A \in |K|\ \forall \psi \in K[O, A]\ ((1_A \otimes t_O)r_A = \psi)$

are fulfilled.

$(K^\bullet; d, t, \nabla, o)$ is called a *diagonal-halfterminal-halfdiagonal-inversional-symmetric monoidal category* (for short *dhth*∇*s-category*; in [6] *dht*∇*-symmetric category*), if $(K^\bullet; d, t, o)$ is a *dhts*-category endowed with a morphism family

$\nabla = (\nabla_A \in K[A \otimes A, A] \mid A \in |K|)$ fulfilling

(D_1^*) $\forall A \in |K|\ (d_A \nabla_A = 1_A)$,

(D_2^*) $\forall A \in |K|\ (\nabla_A d_A d_{A \otimes A} = d_{A \otimes A}(\nabla_A d_A \otimes 1_{A \otimes A}))$. ■

The zero morphisms $o_{A,B}$ absorb all other morphisms at composition and $\otimes$-operation in any *dhts*-category. Because of (o1) and (o2), the unit morphism 1_O is identical with the zero morphism $o_{O,O}$.

The category <u>*Par*</u> of all partial functions between arbitrary sets is an example for a *dhth*∇*s*-category.

In view of the properties of the category $\underline{Par}$ we only will consider $dhth\nabla s$-categories fulfilling the conditions

(O3) $\forall A,\ B \in |K|\ (A \otimes B = O \Rightarrow (A = O\ \vee\ B = O)),$

(o3) $\forall A,\ B,\ C,\ D \in |K|\ \forall \varphi \in K[A, B]\ \forall \psi \in K[C, D]$

$$(\varphi \otimes \psi = o_{A\otimes C, B\otimes D} \Rightarrow (\varphi = o_{A,B}\ \vee\ \psi = o_{C,D})).$$

Remark that $(K^\bullet; d)$ is a ds-category for each $dhts$- category $(K^\bullet; d, t, o)$ and ∇ is the only family in a $dhth\nabla s$-category with the properties (D_1^*) and (D_2^*), cf. [3].

The class $T_K := \{\varphi \in K \mid \varphi t_{codom\varphi} = t_{dom\varphi}\}$ forms a dts-subcategory $\underline{T}_K$ of a $dhts$-category $\underline{K} := (K^\bullet; d, t, o)$ and $(A \otimes B,\ p_1^{A,B} := (1_A \otimes t_B)r_A,\ p_2^{A,B} := (t_A \otimes 1_B)l_B)$ is a categorical product in $\underline{T}_K$, but not in the whole category $\underline{K}$. The morphisms $p_1^{A,B}$ and $p_2^{A,B}$ are called the *canonical projections concerning A and B* ([2]).

The relation $\leq$ defined by

$$\varphi \leq \psi :\Leftrightarrow \exists A, A' \in |K|\ \left(\varphi, \psi \in K[A, A']\ \wedge\ \varphi = d_A(\varphi \otimes \psi)p_2^{A',A'}\right)$$

is a partial order relation and it is compatible with composition and $\otimes$-operation of morphisms (see [3]).

The following conditions are equivalent in any $dhts$-category (see [4]):

$$\varphi = d_A(\varphi \otimes \psi)p_2^{A',A'},$$

$$\varphi = d_A(\psi \otimes \varphi)p_1^{A',A'},$$

$$\varphi d_{A'} = d_A(\varphi \otimes \psi),$$

$$\varphi d_{A'} = d_A(\psi \otimes \varphi).$$

Moreover, each $dhth\nabla s$-category has the properties

$(\mathrm{h}\nabla_1)$ $\forall A, A' \in |K|\ \forall \varphi \in K[A, A']\ (\nabla_A \varphi d_{A'} = d_{A\otimes A}(\nabla_A \varphi \otimes (\varphi \otimes \varphi)\nabla_{A'})),$

(hT_1) $\forall A, A' \in |K|\ \forall \varphi \in K[A, A']\ (\varphi t_{A'} d_I = d_A(\varphi t_{A'} \otimes t_A)),$

therefore $\nabla_A \varphi \leq (\varphi \otimes \varphi)\nabla_{A'}$ and $\varphi t_{A'} \leq t_A$ for all morphisms $\varphi \in K[A, A']$ and all objects A, $A' \in |K|$.

Each morphism set $K[A, B]$ of a $dhth\nabla s$-category $\underline{K}$ forms a meet-semilattice with respect to $\varphi \wedge \psi = d_A(\varphi \otimes \psi)\nabla_B$. This semilattice has the minimum $o_{A,B}$, maximal elements are the total functions. Especially, the morphism sets $K[A, I]$ possess a maximum, namely t_A.

The basic morphisms related to the distingushed object I in any symmetric monoidal category, any $dhts$-category, or even any $dhth\nabla s$-category have some interesting properties as follows:

Lemma 1.2. *Let $K^\bullet$ be a symmetric monidal category. Then the following equalities hold:*

$$r_I = l_I \ ([3]),\ \ a_{I,I,I} = r_I^{-1} \otimes r_I,\ \ b_{A,I,I,B} = 1_{A\otimes I} \otimes 1_{I\otimes B} \ ([6]),\ \ s_{I,I} = 1_{I\otimes I}.$$

Moreover, every $dhts$-category $\underline{K}$ has in addition the properties

$$d_I = r_I^{-1},\ \ r_I d_I = 1_{I\otimes I},\ \ t_I = 1_I \ ([3]),\ \ t_{I\otimes I} = r_I,$$

$$i \in \mathrm{iso}_K[I, I] \Rightarrow i = t_I,\ \ \forall X \in |K|\ \forall x \in K[I, X]\ (x \in \mathrm{iso}K \Rightarrow x^{-1} = t_X).$$

Finally, if $\underline{K}$ is a $dhth\nabla s$-category, then the additional property

$$\nabla_I = r_I$$

is true.

Proof. The identity $a_{A,I,B}(r_A \otimes 1_B) = 1_A \otimes l_B$ is one of the defining properties of monoidal-symmetric categories, hence $a_{I,I,I}(r_I \otimes 1_I) = 1_I \otimes r_I$ by $r_I = l_I$ and $a_{I,I,I} = (r_I^{-1} \otimes r_I)$, since all right-identity morphisms are isomorphisms.

$s_{A,I} l_A = r_A$ is a further defining identity, hence $s_{I,I} l_I = r_I = l_I$ and therefore $s_{I,I} = 1_{I\otimes I}$, because l_I is an isomorphism in K.

In any $dhts$-category one has the defining identity $d_A(1_A \otimes t_A) r_A = 1_A$, hence $1_I = d_I(1_I \otimes t_I) r_I = d_I(1_I \otimes 1_I) r_I = d_I r_I$ since $t_I = 1_I$, consequently $d_I = r_I^{-1}$ and $r_I d_I = 1_{I\otimes I}$.

Each coretraction $\varphi \in K[A, B]$ of a $dhts$-category has the property $\varphi t_B = t_A$. Because d_I is even an isomorphism, one observes $d_I t_{I\otimes I} = t_I = 1_I$, therefore $t_{I\otimes I} = 1_{I\otimes I} t_{I\otimes I} = r_I d_I t_{I\otimes I} = r_I 1_I = r_I$.

One of the characterizing conditions of the diagonal inversions in a $dhth\nabla s$-category is $d_A \nabla_A = 1_A$. Therefore, $\nabla_I = 1_{I\otimes I} \nabla_I = r_I d_I \nabla_I = r_I$ as above.

Now let $i \in K[I,I]$ be an isomorphism of a *dhts*-category $\underline{K}$. Then $i = i1_I = it_I = t_I$, because of $1_i = t_I$.

Let $x \in K[I,X]$ be an isomorphism in a *dhts*-category $\underline{K}$. Then one obtains by the same manner as above $1_I = t_I = xt_X$, hence the assertion. ■

J. Schreckenberger introduced in [3] the important concept of a subidentity in any *dhts*-category $\underline{K}$ in the following way: A morphism $e \in K$ is called *subidentity of an object* $A \in |K|$, if $e \leq 1_A$ with respect to the canonical order relation in $\underline{K}$. The set of all subidentities of an object $A \in |K|$ will be denoted by $E_K(A)$, i.e.

$$E_K(A) := \{e \in K[A,A] \mid e \leq 1_A\}.$$

Each morphism $\varphi \in K[A,A']$ determines in a natural manner a subidentity

$$\alpha(\varphi) := d_A(1_A \otimes \varphi)p_1^{A,A'}, \text{ the subidentity of } \varphi.$$

Subidenties possess a lot of important properties as follows ([3], [9]):

Theorem 1.3. *Let* $\underline{K} = (K^\bullet, d, t, \nabla, o)$ *be a dhth*∇*s-category. Then the following claims hold:*

(E1) $\forall A \in |K|\ \forall e \in E_K(A)\ (ee = e)$,

(E2) $\forall A \in |K|\ \forall e_1, e_2 \in E_K(A)\ (e_1e_2 = e_2e_1)$,

(E3) $\forall A \in |K|\ \forall e_1, e_2 \in E_K(A)\ (e_1e_2 = \inf\{e_1, e_2\})$, *and*

(E4) $\forall A \in |K|\ \forall e_1, e_2 \in E_K(A)\ (e_1 \leq e_2 \Leftrightarrow e_1e_2 = e_1)$, *i.e.*

the set $E_K(A)$ *forms together with the morphism composition a meet-semilattice with maximal element* 1_A *and minimal element* $o_{A,A}$ *related to the canonical partial order relation for each* $A \in |K|$.

Subidenties related to arbitrary morphisms of K *have the following properties:*

(α1) $\forall A, A' \in |K|\ \forall \varphi \in K[A, A]\ (\alpha(\varphi) := d_A(1_A \otimes \varphi)p_1^{A,A'}$
$= d_A(\varphi \otimes 1_A)p_2^{A',A} \leq 1_A)$,

(α2) $\forall A, A' \in |K|\ \forall \varphi \in K[A, A]\ (\alpha(\varphi)\varphi = \varphi)$,

(α3) $\forall A, A' \in |K|\ \forall \varphi \in K[A, A]\ (\alpha(\varphi)t_A = \varphi t_{A'})$,

(α4) $\forall A, A' \in |K|\ \forall \varphi \in K[A, A]\ (\alpha(\varphi) = 1_A \Leftrightarrow \varphi t_{A'} = t_A)$,

(α5) $\forall A, A' \in |K|\ \forall \varphi \in K[A, A]\ (\alpha(\varphi) = o_{A,A} \Leftrightarrow \varphi = o_{A,A'})$,

(α6) $\forall A \in |K|\ \forall e \in E_K(A)\ (\alpha(e) = e)$,

(α7) $\forall A, A' \in |K|\ \forall \varphi \in K[A, A']\ \forall e \in E_K(A)\ (e\alpha(\varphi) = \alpha(e\varphi) \leq e)$,

(α8) $\forall A, A' \in |K|\ \forall \varphi \in K[A, A']\ \forall e \in E_K(A)\ (e\varphi = \varphi \Leftrightarrow \alpha(\varphi) \leq e)$,

(α9) $\forall A, A' \in |K|\ \forall \varphi \in K[A, A']\ \forall e \in E_K(A)\ (e \leq \alpha(\varphi) \Rightarrow \alpha(e\varphi) = e)$,

(α10) $\forall A, A' \in |K|\ \forall \varphi, \psi \in K[A, A']\ (\varphi \leq \psi \Leftrightarrow \alpha(\varphi)\psi = \varphi)$,

(α11) $\forall A, A' \in |K|\ \forall \varphi, \psi \in K[A, A']\ (\varphi \leq \psi \Rightarrow \alpha(\varphi) \leq \alpha(\psi))$,

(α12) $\forall A, A' \in |K|\ \forall \varphi, \psi, \xi \in K[A, A']$
$(\varphi \leq \xi\ \wedge\ \psi \leq \xi\ \wedge\ \alpha(\varphi) \leq \alpha(\psi) \Rightarrow \varphi \leq \psi)$,

(α13) $\forall A, A' \in |K|\ \forall \varphi, \psi \in K[A, A']\ (\alpha(\varphi) = \alpha(\psi)\ \wedge$
$\exists \xi\ (\varphi \leq \xi \wedge \psi \leq \xi) \Rightarrow \varphi = \psi)$,

(α14) $\forall A, A' \in |K|\ \forall \varphi, \psi \in K[A, A']\ (\varphi \leq \psi\ \wedge\ \alpha(\varphi) = \alpha(\psi) \Rightarrow \varphi = \psi)$,

(α15) $\forall A, A', B \in |K|\ \forall \varphi \in K[A, A']\ \forall \psi \in K[A', B]\ (\alpha(\varphi\psi) \leq \alpha(\varphi))$,

(α16) $\forall A, A', B \in |K|\ \forall \varphi \in K[A, A']\ \forall \psi \in K[A', B]\ (\varphi\alpha(\psi) = \alpha(\varphi\psi)\varphi)$,

(α17) $\forall A, A', B \in |K|\ \forall \varphi \in K[A, A']\ \forall \psi \in K[A', B]\ (\alpha(\varphi\psi) = \alpha(\varphi\alpha(\psi)))$,

(α18) $\forall A, A', B, B' \in |K|\ \forall \varphi \in K[A, A']\ \forall \varphi' \in K[B, B']$
$(\alpha(\varphi \otimes \varphi') = \alpha(\varphi) \otimes \alpha(\varphi'))$,

(α19) $\forall A, A_1, A_2 \in |K|\ \forall \varphi_i \in K[A, A_i]\ (i = 1, 2)$
$(\alpha(d_A(\varphi_1 \otimes \varphi_2)) = \alpha(\varphi_1)\alpha(\varphi_2))$,

(α20) $\forall A, A_1, A_2 \in |K|\ \forall \varphi \in K[A, A_1]\ \forall \psi \in T_K[A_1, A_2]\ \forall \chi \in T_K[A, A_2]$
$(\alpha(\varphi) = \alpha(d_A(\varphi \otimes \chi)) = \alpha(d_A(\varphi\psi \otimes \chi)))$,

(α21) $\forall A, A' \in |K|\ \forall \varphi_1, \varphi_2 \in K[A, A']$
$(\alpha(\varphi_1)\varphi_2 = \varphi_2 \Leftrightarrow \alpha(\varphi_2) \leq \alpha(\varphi_1))$,

(α22) $\forall A \in |K|\ (\alpha(\nabla_A) = \alpha(\nabla_A d_A) = \nabla_A d_A)$,

(α23) $\forall A, A' \in |K|\ \forall \varphi_1, \varphi_2 \in K[A, A']$
$(\alpha((\varphi_1 \otimes 1_{A'})\nabla_{A'}) = \alpha((\varphi_2 \otimes 1_{A'})\nabla_{A'}) \Leftrightarrow \varphi_1 = \varphi_2)$,

(α24) $\forall A, A' \in |K|\ \forall \varphi_1, \varphi_2 \in K[A, A']$
$(\alpha(\varphi_2)\varphi_1 = \alpha(\varphi_1)\varphi_2 \Rightarrow d_A(\varphi_1 \otimes \varphi_2)\nabla_{A'} = d_A(\varphi_1 \otimes \varphi_2)p_i^{A',A'}, (i = 1, 2))$,

(α25) $\forall A, A' \in |K|\ \forall \varphi_1, \varphi_2 \in K[A, A']$
$(\alpha(\varphi_1)\varphi_2 = d_A(\varphi_1 \otimes \varphi_2)\nabla_{A'} \Leftrightarrow \alpha(\varphi_2)\varphi_1 = d_A(\varphi_1 \otimes \varphi_2)\nabla_A)$,

(α26) $\forall A, A' \in |K|\ \forall \varphi \in K[A, A']\ (\alpha(\nabla_A \varphi) = \nabla_A \varphi d_A)$,

(α27) $\forall A, A' \in |K|\ \forall \varphi \in K[A, A']\ \forall \varphi^* \in K[A', A]$
$(\alpha(\varphi) = \varphi\varphi^* \Leftrightarrow (\varphi\varphi^*\varphi = \varphi\ \wedge\ \varphi\varphi^* \leq 1_A))$. ■

2. Monoidal functors

In applications to theories of algebraic structures, functors $F : \underline{K} \to \underline{K'}$ between $dhth\nabla s$-categories are of interest which preserve in addition to the functor properties the $dhth\nabla s$-structure with respect to a family $\widetilde{F} = \left(\widetilde{F}\langle X, Y\rangle \mid X, Y \in |K|\right)$ of isomorphisms $\widetilde{F}\langle X, Y\rangle : XF \otimes YF \to (X \otimes Y)F$

in $\underline{K'}$ and an isomorphism i_F between I' and IF, where I and I' are the distinguished objects in $\underline{K}$ and $\underline{K'}$, respectively, ([2], [4], [8]).

Definition 2.1 ([8]). A functor $F : K^\bullet \to K'^\bullet$ between symmetric monoidal categories $K^\bullet$ and $K'^\bullet$ is called *monoidal with respect to a family of morphisms*

$$\widetilde{F} = \Big(\widetilde{F}\langle X, Y\rangle : XF \otimes YF \to (X \otimes Y)F \mid X, Y \in |K|\Big) \text{ of } K'$$

and to a morphism

$$i_F : I' \to IF,$$

for short $(F, \widetilde{F}, i_F) : K^\bullet \to K'^\bullet$, iff the following conditions are fulfilled:

$(F\sim)$ $\forall X, Y \in |K| \ \Big(\widetilde{F}\langle X, Y\rangle \in \mathrm{iso}(K')\Big)$,

(FI) $i_F \in \mathrm{iso}(K')$,

(FA) $\forall X, Y, Z \in |K|\Big(\Big(1'_{XF} \otimes \widetilde{F}\langle Y, Z\rangle\Big)\widetilde{F}\langle X, Y \otimes Z\rangle(a_{X,Y,Z}F) =$

$= a'_{XF,YF,ZF}\Big(\widetilde{F}\langle X, Y\rangle \otimes 1'_{ZF}\Big)\widetilde{F}\langle X \otimes Y, Z\rangle\Big)$,

(FR) $\forall X \in |K| \ \Big(\widetilde{F}\langle X, I\rangle(r_X F) = \Big(1_{XF} \otimes i_F^{-1}\Big)r'_{XF}\Big)$,

(FS) $\forall X, Y \in |K| \ \Big(\widetilde{F}\langle X, Y\rangle(s_{X,Y}F) = s'_{XF,YF}\widetilde{F}\langle Y, X\rangle\Big)$,

(FM) $\forall \varphi : X \to Y, \forall \psi : U \to V \in K$
$\Big((\varphi F \otimes \psi F)\widetilde{F}\langle Y, V\rangle = \widetilde{F}\langle X, U\rangle(\varphi \otimes \psi)F\Big)$. ■

Corollary 2.2. *Let* $(F, \widetilde{F}, i_F) : K^\bullet \to K'^\bullet$ *be a monoidal functor between symmetric monoidal categories. Then*

(FL) $\forall X \in |K| \ \Big(\widetilde{F}\langle I, X\rangle(l_X F) = \Big(i_F^{-1} \otimes 1_{XF}\Big)l'_{XF}\Big)$.

Proof. The validity of (FL) is a consequence of (FR) and (FS) by the properties of symmetric monoidal categories in the following way:

$$\begin{aligned}
\widetilde{F}\langle I,X\rangle(l_XF) &= s'_{IF,XF}\widetilde{F}\langle X,I\rangle(s_{I,X}F)^{-1}(l_XF) && \text{((FS))}\\
&= s'_{IF,XF}\widetilde{F}\langle X,I\rangle((s_{X,I}l_X)F) && (s^{-1}_{A,B}=s_{B,A})\\
&= s'_{IF,XF}\widetilde{F}\langle X,I\rangle(r_XF) && (s_{IX}l_X=r_X)\\
&= s'_{IF,XF}(1'_{XF}\otimes i_F^{-1})r'_{XF} && \text{((FR))}\\
&= (i_F^{-1}\otimes 1'_{XF})s'_{I',XF}r'_{XF}\\
&= (i_F^{-1}\otimes 1'_{XF})l'_{XF}. && \blacksquare
\end{aligned}$$

Definition 2.3 ([8]). A monoidal functor $(F,\widetilde{F},i_F):\underline{K}\to\underline{K'}$ between *ds*-categories $\underline{K}$ and $\underline{K'}$ is called *d-monoidal*, if in addition the condition

$$\text{(FD)}\quad \forall A\in|K|\ \Big(d_AF=d'_{AF}\widetilde{F}\langle A,A\rangle\Big)$$

holds. ■

Obviously, the identical functor of $K^\bullet$ forms a monoidal functor

$$\Big(1_K,\ \Big(\widetilde{1}_K\langle X,Y\rangle=1_{XF\otimes YF}\mid X,Y\in|K|\Big),\ i_{1_K}=1_I\Big):K^\bullet\to K^\bullet$$
$$(X\mapsto X,\ \varphi\mapsto\varphi)$$

and the constant functor from $K^\bullet$ into $K'^\bullet$ too,

$$\Big(E,\ \Big(\widetilde{E}\langle X,Y\rangle=1'_{I'}\mid X,Y\in|K|\Big),i_E=1'_{I'}\Big):K^\bullet\to K'^\bullet\ (X\mapsto I',\ \varphi\mapsto 1'_{I'}),$$

where $K^\bullet$ and $K'^\bullet$ are arbitrary symmetric monoidal categories.

Moreover:

Proposition 2.4. *Let $\underline{K}$ and $\underline{K'}$ be dhts-categories such that there are the distinguished zero-objects $O\in|K|,\ O'\in|K'|$. Then*

$$E_0 : \underline{K} \to \underline{K}', \textit{ defined by } X \mapsto \begin{cases} I' & \text{if} \quad X \neq O, \\ O' & \text{if} \quad X = O, \end{cases} \quad \textit{and}$$

$$(\varphi : X \to Y) \mapsto \begin{cases} 1'_{I'} & \text{if} \quad X \neq O \ \wedge \ Y \neq O \ \wedge \ \varphi \neq o_{X,Y}, \\ o'_{I',I'} & \text{if} \quad X \neq O \ \wedge \ Y \neq O \ \wedge \ \varphi = o_{X,Y}, \\ t'_{O'} & \text{if} \quad X = O \ \wedge \ Y \neq O, \\ o' & \text{if} \quad X \neq O \ \wedge \ Y = O, \\ 1'_{O'} & \text{if} \quad X = Y = O, \end{cases}$$

is d-monoidal with respect to

$$\widetilde{E_0}\langle X, Y\rangle = \begin{cases} r'_{I'} & \text{if } X \neq O \neq Y, \\ 1'_{O'} & \text{otherwise}, \end{cases} \qquad \text{and} \quad i_{E_0} = 1'_{I'} \ .$$

Proof. The functor properties are easy to verify by consideration of the separate cases. By definition, all morphisms $E_0\langle X, Y\rangle$ are isomorphisms and i_{E_0} is an isomorphism too.

Ad (FA): If X, Y, and Z all are different from O, then

$$\left(1'_{XE_0} \otimes \widetilde{E_0}\langle Y, Z\rangle\right) \widetilde{E_0}\langle X, Y \otimes Z\rangle(a_{X,Y,Z}E_0) = (1'_{I'} \otimes r'_{I'})r'_{I'}1'_{I'} = (1'_{I'} \otimes r'_{I'})l'_{I'}$$

$$= l'_{I'\otimes I'}r'_{I'} = a'_{I',I',I'}(l'_{I'} \otimes 1'_{I'})r'_{I'} = a'_{I',I',I'}(r'_{I'} \otimes 1'_{I'})r'_{I'}$$

$$= a'_{XE_0,YE_0,ZE_0}\left(\widetilde{E_0}\langle X, Y\rangle \otimes 1'_{ZE_0}\right) \widetilde{E_0}\langle X \otimes Y, Z\rangle.$$

On the other hand:

$$X = O \ \vee \ Y = O \ \vee \ Z = O$$

$$\Rightarrow X \otimes (X \otimes Z) = O = (X \otimes Y) \otimes Z$$

$$\Rightarrow a_{X,Y,Z}E_0 = o_{O,O}E_0 = o'_{O',O'} \ \wedge \ a'_{XE_0,YE_0,ZE_0} = o'_{O',O'}$$

$$\Rightarrow \left(1'_{XE_0} \otimes \widetilde{E_0}\langle Y,Z\rangle\right) \widetilde{E_0}\langle X, Y\otimes Z\rangle(a_{X,Y,Z}E_0) = o'_{O',O'} \ \wedge$$

$$a'_{XE_0,YE_0,ZE_0}\left(\widetilde{E_0}\langle X,Y\rangle \otimes 1'_{ZE_0}\right) \widetilde{E_0}\langle X\otimes Y, Z\rangle = o'_{O',O'}.$$

Ad (FR): Because $i_{E_0} \in K'[I', IE_0]$ is an isomorphism in K' and $IE_0 = I'$, one has $i_{E_0} = 1'_{I'}$ by Lemma 1.2. Let $X \neq O$. Then

$$\widetilde{E_0}\langle X,I\rangle(r_XE_0) = r'_{I'}1'_{I'} = \left(1'_{I'} \otimes 1'_{I'}\right)r'_{I'} = \left(1'_{I'} \otimes i_{E_0}^{-1}\right)r'_{XE_0}.$$

Otherwise, $X = O$ implies $XE_0 = O' \ \wedge \ X \otimes I = O \otimes I = O$, hence $\widetilde{E_0}\langle X,I\rangle = \widetilde{E_0}\langle O,I\rangle = 1'_{O'} = o'_{O',O'} \ \wedge \ r_XE_0 = 1_OE_0 = 1'_{O'} = o'_{O',O'} = r'_{O'} = r'_{XE_0}$. Therefore

$$\begin{aligned}\widetilde{E_0}\langle X,I\rangle(r_XE_0) = 1'_{O'}1'_{O'} = 1'_{O'} = o'_{O',O'} &= (o'_{O',O'} \otimes i_{E_0}^{-1})o'_{O',O'} \\ &= (1'_{XE_0} \otimes i_{E_0}^{-1})r'_{XE_0}.\end{aligned}$$

Ad (FS): Since $s'_{I',I'} = 1'_{I'\otimes I'}$, we have

$$\widetilde{E_0}\langle X,Y\rangle(s_{X,Y}E_0) = r'_{I'}1'_{I'} = s'_{I',I'}r'_{I'} = s'_{XE_0,YE_0}\widetilde{E_0}\langle Y,X\rangle$$

for $X \neq O \neq Y$.

Assuming $X = O$ or $Y = O$ one obtains

$$X \otimes Y = O = Y \otimes X \ \text{ and } \ \widetilde{E_0}\langle X,Y\rangle = 1'_{O'} = o'_{O',O'} = \widetilde{E_0}\langle Y,X\rangle,$$

hence $\widetilde{E_0}\langle X,Y\rangle(s_{X,Y}E_0) = o'_{O',O'} = s'_{XE_0,YE_0}\widetilde{E_0}\langle Y,X\rangle$.

Ad (FM): For $O \notin \{X,Y,U,V\} \ \wedge \ \varphi \neq o_{X,Y} \ \wedge \ \psi \neq o_{U,V}$ one has $\varphi \otimes \psi \neq o_{X\otimes U, Y\otimes V}$, therefore

$$(\varphi E_0 \otimes \psi E_0)\widetilde{E_0}\langle Y,V\rangle = (1'_{I'} \otimes 1'_{I'})r'_{I'} = r'_{I'}1'_{I'} = \widetilde{E_0}\langle X,U\rangle((\varphi\otimes\psi)E_0).$$

In the case $O \in \{X,Y,U,V\}$ one obtains

$$\varphi \otimes \psi = o_{X\otimes U, Y\otimes V} \ \wedge \ (\varphi \otimes \psi)E_0 = o'_{(X\otimes U)E_0,(Y\otimes V)E_0} \ \wedge$$

$$\varphi E_0 \otimes \psi E_0 = o'_{XE_0\otimes UE_0, YE_0\otimes VE_0}, \text{ hence}$$

$$\begin{aligned}(\varphi E_0 \otimes \psi E_0)\widetilde{E_0}\langle Y,V\rangle &= o'_{XE_0\otimes UE_0,YE_0\otimes VE_0}\widetilde{E_0}\langle Y,V\rangle = o'_{XE_0\otimes UE_0,(Y\otimes V)E_0} \\ &= \widetilde{E_0}\langle X,U\rangle o'_{(X\otimes U)E_0,(Y\otimes V)E_0} = \widetilde{E_0}\langle X,U\rangle((\varphi\otimes\psi)E_0).\end{aligned}$$

Ad (FD): The assumption $X \neq O$ yields directly

$$d_X E_0 = 1'_{I'} = d'_{I'} r'_{I'} = d'_{XE_0}\widetilde{E_0}\langle X,X\rangle.$$

For $X = O$ one has:

$$d_X E_0 = o_{O,O}E_0 = o'_{O',O'} = d'_{XE_0}\widetilde{E_0}\langle X,X\rangle. \qquad \blacksquare$$

Each d-monoidal functor $(F,\widetilde{F},i_F)$ between $dhts$-categories possesses the following properties (see [3], [8]):

(FI*) $t'_{IF} = i_F^{-1}$,

(Fmon) $\forall \varphi,\, \psi \in K \ (\varphi \leq \psi \Rightarrow \varphi F \leq \psi F)$,

(FT) $\forall X \in |K| \ (t_X F\, t'_{IF} = t'_{XF})$,

(FP) $\forall X,Y \in |K| \ (p_j^{X,Y} F = \left(\widetilde{F}\langle X,Y\rangle\right)^{-1} p'^{XF,YF}_j \ ; \ \ j = 1,2)$,

(FE) $\forall A \in |K| \ (e \leq 1_A \Rightarrow eF \leq 1_{AF})$,

(FEα) $\forall X,\, Y \in |K| \ \forall \varphi \in K[X,Y] \ ((\alpha(\varphi))F = \alpha(\varphi F))$.

Let $\underline{K}$, $\underline{K'}$ be $dhth\nabla s$-categories and let $(F,\widetilde{F},i_F) : \underline{K} \to \underline{K'}$ be a d-monoidal functor. Then in addition the following properties hold ([8]):

(Finf) $\forall X,Y \in |K| \ \forall \varphi,\psi \in K[X,Y]$

$((d_X(\varphi\otimes\psi)\nabla_Y)F = d'_{XF}(\varphi F \otimes \psi F)\nabla'_{YF})$,

(Finj) $\forall X,Y \in |K| \ \forall \varphi \in K[X,Y]$

$$((\varphi \otimes \varphi)\nabla_Y = \nabla_X \varphi \Rightarrow (\varphi F \otimes \varphi F)\nabla'_{YF} = \nabla'_{XF}(\varphi F)),$$

$$(F\nabla) \quad \forall X \in |K| \; (\nabla_{XF} = \widetilde{F}\langle X, X\rangle \nabla'_X F),$$

$$(F\nabla_1) \quad \forall X, Y, U \in |K| \; \forall \varphi \in K[X, U] \; \forall \psi \in K[Y, U]$$

$$(((\varphi \otimes \psi)\nabla_U)F = \widetilde{F}\langle X, Y\rangle((\varphi \otimes \psi)F)\nabla'_{UF}),$$

$$(F\nabla_2) \quad \forall X, Y \in |K| \; \forall \varphi.\psi \in K[X, Y]$$

$$((\varphi \otimes \psi)\nabla_Y = \nabla_X \varphi \Rightarrow (\varphi F \otimes \psi F)\nabla'_{YF} = \nabla'_{XF}(\varphi F)).$$

Obviously, property (Finj) is a special case of (F∇_2) and this property expresses once more the monotony of the functor F, namely $\varphi \leq \psi \Rightarrow \varphi F \leq \psi F$.

The so-called *zero functor* $Z : \underline{K} \to \underline{K'}$ is defined by $XZ = O'$ for all objects $X \in |K|$ and $\varphi Z = 1'_{O'}$ for all morphisms $\varphi \in K$. Trivially, this functor is a d-monoidal one.

Definition 2.5 ([8]). A d-monoidal functor $(F, \widetilde{F}, i_F)$ between *dhts*-categories will be called *dht-monoidal functor,* iff either $F = Z$ or, besides the conditions of a d-monoidal functor, the condition

$$\text{(FZ)} \qquad OF = O' \;\wedge\; \forall X \in |K| \; (XF = O' \Rightarrow X = O)$$

is fulfilled. ■

Proposition 2.6 ([8]). *Let* $(F, \widetilde{F}, i_F) : \underline{K} \to \underline{K'}$ *be a dht-monoidal functor such that* $F \neq Z$. *Then one obtains:*

$$\forall X \in |K| \left(\widetilde{F}\langle X, O\rangle = \widetilde{F}\langle O, X\rangle = 1'_{O'}\right),$$

$$\forall X, Y \in |K| \; (o_{X,Y} F = o'_{XF,YF}),$$

$$oF = t'_{IF} o' \quad (\Leftrightarrow o' = i(oF)). \qquad ■$$

By the structure of *dhts*-categories $\underline{K}$ and $\underline{K'}$, each functor $F : \underline{K} \to \underline{K'}$ determines with respect to arbitrary objects $X, Y \in |K|$ the morphisms

$$F^*\langle X, Y\rangle := d'_{(X\otimes Y)F}\left(p_1^{X,Y} F \otimes p_2^{X,Y} F\right) \in K'[(X \otimes Y)F, XF \otimes YF]$$

in the category K'.

In the case that $(F, \widetilde{F}, i_F) : \underline{K} \to \underline{K'}$ is a d-monoidal functor, the morphisms $\widetilde{F}\langle X, Y\rangle$ are uniquely determined by

$$\left(\widetilde{F}\langle X, Y\rangle\right)^{-1} = d'_{(X\otimes Y)F}\left(p_1^{X,Y}F \otimes p_2^{X,Y}F\right) = F^*\langle X, Y\rangle$$

(see [2]).
Moreover:

Theorem 2.7 (see [8]). *Assume that $F : \underline{K} \to \underline{K}'$ is any functor from a dht-symmetric category $\underline{K}$ into a dht-symmetric category $\underline{K}'$ satisfying the following conditions:*

(F$*$) $\quad \forall X, Y \in |K|(F^*\langle X, Y\rangle \in \text{ iso } (K')),$

(FI*) $\quad {t'}_{IF} \in \text{iso}(K'),$

(FM*) $\quad \forall \varphi, \psi \in |K| \; ((\varphi \otimes \psi)F\, F^*\langle X', Y'\rangle = F^*\langle X, Y\rangle(\varphi F \otimes \psi F)).$

Then $(F, \widetilde{F}, i_F) : \underline{K} \to \underline{K}'$ is d-monoidal with $\widetilde{F}\langle X, Y\rangle := (F^*\langle X, Y\rangle)^{-1}$, $i_F := {t'}_{IF}^{-1}$. ■

3. PROPERTIES OF THE *Hom*-FUNCTORS

Any *dts*-category contains not necessarily an initial object O. For *dts*-categories $\underline{K}$ without initial objects one has the following fact.

Theorem 3.1. *Let $\underline{K}$ be a dts-category and let H^A be the usual Hom-functor from the underlying category K into the category Set with reference to any object $A \in |K|$. Then H^A is a d-monoidal functor related to $\widetilde{H^A}$ and i_{H^A}, where*

$$\widetilde{H^A} = \left(\widetilde{H^A}\langle X, Y\rangle : K[A, X] \times K[A, Y] \to K[A, X \otimes Y] \mid X, Y \in |K|\right),$$

defined by

$$(u_1, u_2) \mapsto \left(\widetilde{H^A}\langle X, Y\rangle\right)((u_1, u_2)) := d_A(u_1 \otimes u_2)),$$

and

$$i_{H^A} : \{\emptyset\} \to K[A, I], \ \textit{defined by } (\emptyset \mapsto t_A).$$

Proof. Let A be any object of the category K. Then the functor properties of H^A are well-known.

Ad (F$\sim$): Each ordered pair $(u_1, u_2) \in K[A, X] \times K[A, Y]$, $X, Y \in |K|$, determines uniquely the morphism $d_A(u_1 \otimes u_2) \in K[A, X \otimes Y]$. Conversely, each morphism $u \in K[A, X \otimes Y]$ determines the both morphisms $up_1^{X,Y} \in K[A, X]$, $up_2^{X,Y} \in K[A, Y]$ and $d_A(u_1 \otimes u_2)p_i^{X,Y} = u_i$ for $i = 1, 2$ shows that $\widetilde{H^A}\langle X, Y\rangle$ is an isomorphism in Set.

Ad (FI): In any *dts*-category, the set $K[A, I]$ consists of one element only, therefore the only total function i_{H^A} from $I^{Set} = \{\emptyset\}$ onto $IH^A = K[A, I]$ is an isomorphism in Set and $i_{H^A}^{-1} = t_{IH^A}^{Set}$.

Ad (FA): Let $u \in XH^A$, $v \in YH^A$, $w \in ZH^A$ be arbitrary morphisms, X, Y, $Z \in |K|$. Then one has

$$\left(a^{Set}_{XH^A,YH^A,ZH^A}\left(\widetilde{H^A}\langle X, Y\rangle \times 1^{Set}_{ZH^A}\right)\widetilde{H^A}\langle X \otimes Y, Z\rangle\right)((u, (v, w)))$$

$$= \left(\left(\widetilde{H^A}\langle X, Y\rangle \times 1^{Set}_{ZH^A}\right)\widetilde{H^A}\langle X \otimes Y, Z\rangle\right)(((u, v), w))$$

$$= \widetilde{H^A}\langle X \otimes Y, Z\rangle((d_A(u \otimes v), w)) = d_A(d_A(u \otimes v) \otimes w)$$

and, on the other hand,

$$\left(\left(1^{Set}_{XH^A} \times \widetilde{H^A}\langle Y, Z\rangle\right)\widetilde{H^A}\langle X, Y \otimes Z\rangle(a_{X,Y,Z}H^A)\right)((u, (v, w)))$$

$$= \left(\widetilde{H^A}\langle X, Y \otimes Z\rangle(a_{X,Y,Z}H^A)\right)((u, d(v \otimes w)))$$

$$= (a_{X,Y,Z}H^A)(d_A(u \otimes d_A(v \otimes w)))$$

$$= d_A(u \otimes d_A(v \otimes w))a_{X,Y,Z} = d_A(1_A \otimes d_A)a_{A,A,A}((u \otimes v) \otimes w)$$

$$= d_A(d_A(u \otimes v) \otimes w),$$

hence

$$a^{Set}_{XH^A,YH^A,ZH^A}\Big(\widetilde{H^A}\langle X,Y\rangle \times 1^{Set}_{ZH^A})\widetilde{H^A}\langle X\otimes Y,Z\rangle\Big)$$
$$= \Big(1^{Set}_{XH^A} \times \widetilde{H^A}\langle Y,Z\rangle\Big)\widetilde{H^A}\langle X,Y\otimes Z\rangle(a_{X,Y,Z}H^A).$$

Ad (FR): The morphism set $K[A,I]$ consists in each *dts*-category of one element t_A only. So one obtains for each morphism $u \in K[A,X] = XH^A$, $X \in |K|$,

$$\Big(\widetilde{H^A}\langle X,I\rangle(r_XH^A)\Big)((u,t_A)) = (r_XH^A)(d_A(u\otimes t_A))$$
$$= d_A(u\otimes t_A)r_X = d_A(1_A\otimes t_A)r_Au = u$$

and

$$\Big(\Big(1^{Set}_{XH^A}\times t^{Set}_{IH^A}\Big)r^{Set}_{XH^A}\Big)((u,t_A)) = r^{Set}_{XH^A}((u,\emptyset)) = u,$$

thus the validity of (FR).

Ad (FS): Since for all $u \in XH^A, v\in YH^A$, X, $Y \in |K|$, the morphisms

$$\Big(\widetilde{H^A}\langle X,Y\rangle(s_{X,Y}H^A\Big)((u,v)) = \Big(s_{X,Y}H^A\Big)(d_A(u\otimes v)) = d_A(u\otimes v)s_{X,Y}$$
$$= d_As_{A,A}(v\otimes u) = d_A(v\otimes u) \text{ and}$$
$$\Big(s^{Set}_{XH^A,YH^A}\widetilde{H^A}\langle Y,X\rangle\Big)((u,v)) = \Big(\widetilde{H^A}\langle Y,X\rangle\Big)((v,u)) = d_A(v\otimes u)$$

coincide, the condition is fulfilled.

Ad (FM): The equation

$$\Big(\Big(\varphi H^A\times\psi H^A\Big)\widetilde{H^A}\langle Y,V\rangle\Big)((u,v)) = \Big(\widetilde{H^A}\langle Y,V\rangle\Big)((u\varphi,v\psi)) = d_A(u\varphi\otimes v\psi)$$
$$= d_A(u\otimes v)(\varphi\otimes\psi) = \Big((\varphi\otimes\psi)H^A\Big)(d_A(u\otimes v))$$
$$= \Big(\widetilde{H^A}\langle X,U\rangle((\varphi\otimes\psi)H^A)\Big)((u,v))$$

is valid for all objects X, Y, U, $V \in |K|$ and all morphisms $\varphi \in K[X,Y]$, $\psi \in K[U,V]$, $u \in K[A,X] = XH^A$, $v \in K[A,U] = UH^A$, therefore (FM) is an identity.

Ad (FD): For each $X \in |K|$ and all $u \in XH^A$ one has

$$\left(d_X H^A\right)(u) = ud_X = d_A(u \otimes u) = \widetilde{H^A}\langle X, X\rangle((u,u)) = \left(d^{Set}_{XH^A}\widetilde{H^A}\langle X, X\rangle\right)(u).$$

■

The argumentation above shows that the proof of some properties only need the d-monoidal-symmetric structure of the category K, so one obtains:

Corollary 3.2. *Let $K^\bullet$ be a ds-category. Then there is to each pair (X,Y) of objects of $K^\bullet$ and each Hom-functor $H^A : K^\bullet \to Set$ a function $\widetilde{H^A}\langle X, Y\rangle$: $K[A,X] \times K[A,Y] \to K[A, X \otimes Y]$ such that the conditions* (FA), (FS), (FM), (FD) *are valid.* ■

Every dts-category contains the distinguished terminal object I, i.e. that the set $K[X,I]$ consists of exatly one element t_X. Conversely, in general there are no information about the sets $K[I,X]$. There are possibly objects X in $\underline{K}$ such that $K[I,X] = \emptyset$. The dts-category $\underline{Set}$ has the property that $Set[I,X]$ contains at least one element for each $X \neq \emptyset$.

Let $(K^\bullet, d)$ be a ds-category containing an initial object O with the property

(O1) $\quad \forall X \in |K| \ (O \otimes X = O = X \otimes O)$,

then every morphism set $K[O,X]$ constists of exactly one element, say z_X. In this case, $(K^\bullet, d)$ has the property

(zz) $\quad \forall A, B, X \in |K|, \forall \varphi \in K[A,B]$
$$(z_A\varphi = z_B \ \wedge \ \varphi \otimes z_X = z_{B\otimes X} \ \wedge \ z_X \otimes \varphi = z_{X\otimes B}).$$

Proposition 3.3. *Let $\underline{K}$ be a ds-category containing an initial object O with the property* (O1)*. Then this object O induces a special d-monoidal functor $H^O : \underline{K} \to \underline{Set}$ as follows:*

$$\forall X \in |K| \ (XH^O = \{z_X\}),$$

$$\forall U, V \in |K| \ \forall \varphi \in K[U,V] \ (\varphi H^O : K[O,U] \to K[O,V] \quad (z_U \mapsto z_U\varphi = z_V)),$$

$$\forall X, Y \in |K| \ \Big(\widetilde{H^O}\langle X, Y\rangle : K[O,X] \times K[O,Y] \to K[O, X \otimes Y] \quad ((z_X, z_Y) \mapsto z_{X\otimes Y})\Big),$$

$$i_{H^O} : \{\emptyset\} \to K[O,I] \, (\emptyset \mapsto z_I).$$

Proof. Since every morphism set $K[O,X]$ consists of exactly one element z_X for each object $X \in |K|$, all sets of the form $K[O,X] \times K[O,Y]$ consist of one element too. Therefore, all functions $\widetilde{H^O}\langle X, Y\rangle$ are isomorphisms, the function i_{H^O} is an isomorphism and all conditions (FA), (FR), (FS), (FM), and (FD) are fulfilled. ■

Remark that generally the Hom-functor H^A is not a d-monoidal functor for arbitrary $dhts$-categories. To each element $(u_1, u_2) \in K[A, X_1] \times K[A, X_2]$ there is in fact allways the morphism $d_A(u_1 \otimes u_2) \in K[A, X_1 \otimes X_2]$, but $d(u_1 \otimes u_2)p_1^{X_1,X_2}$ has not to be equal to u_i $(i = 1, 2)$ in general, therefore, $\widetilde{H^A}\langle X_1, X_2\rangle$ must not be an isomorphism in Par. The concept of a subidenty in $dhts$-categories introduced by J. Schreckenberger ([3]) allows a modification of the concept of a Hom-functor such that one obtains a d-monoidal functor.

4. Functors defined by subidentities

Theorem 4.1 ([3]). *Let $\underline{K}$ be a dhts-category. Each subidentity $e \leq 1_A$ in $\underline{K}$ determines a d-monoidal functor $(H^e, \widetilde{H}^e, i_{H^e}) : \underline{K} \to \underline{Par}$ by*

(1) $XH^e := \{u \in K[A,X] \mid \alpha(u) = e\}$, $X \in |K|$,

(2) $(\varphi : X \to Y)H^e := \varphi H^e : XH^e \to YH^e (\in Par)$, *defined by*

$$(\varphi H^e)(u) := u\varphi \text{ for } u \in D(\varphi H^e) := \{u \in XH^e \mid u\alpha(\varphi) = u\},$$

(3) $\widetilde{H}^e\langle X, Y\rangle : XH^e \times YH^e \to (X \otimes Y)H^e$, *defined by*

$$(u_1, u_2) \mapsto \widetilde{H}^e\langle X, Y\rangle((u_1, u_2)) := d_A(u_1 \otimes u_2),$$

(4) $i_{H^e} : \{\emptyset\} = I^{Par} \to IH^e = \{u \in K[A, I] \mid \alpha(u) = e\}$.

Proof. Obviosly, $u\varphi \in K[A, Y]$ and $\alpha(u\varphi) = \alpha(u\alpha(\varphi)) = \alpha(u) = e$ (with respect to (α17) and (2)) shows $u\varphi \in X'H^e$.

$1_X H^e$ is the identical function id_{XH^e} of XH^e since $D(1_X H^e) = \{u \in XH^e \mid u1_X = u\} = XH^e$ and $(1_X H^e)(u) = u1_X = u$ for all $u \in XH^e$, i.e. $1_X H^e = id_{XH^e}$.

Let $\varphi \in K[X, Y]$, $\psi \in K[Y, Z]$ be arbitrary morphisms in K. Then, by the properties of the subidenties and the defining conditions above, $(\varphi\psi)H^e = (\varphi H^e)(\psi H^e)$ because of

$$u \in D((\varphi\psi)H^e) \Rightarrow \alpha(u) = e \ \wedge \ u\alpha(\varphi\psi) = u$$

$$\Rightarrow u\alpha(\varphi) = \alpha(u\varphi)u = \alpha(u\alpha(\varphi\psi)\varphi)u = \alpha(u\alpha(\varphi\psi)\alpha(\varphi))u$$

$$= \alpha(u\alpha(\varphi\psi))u = \alpha(u)u = u$$

$$\wedge \ u\varphi\alpha(\psi) = u\alpha(\varphi\psi)\varphi = u\varphi \ \wedge \ \alpha(u\varphi) = \alpha(u\alpha(\varphi)) = \alpha(u) = e$$

$$\Rightarrow u \in D(\varphi H^e) \ \wedge \ u\varphi \in D(\psi H^e) \Rightarrow u \in D((\varphi H^e)(\psi H^e));$$

$$u \in D((\varphi H^e)(\psi H^e)) \Rightarrow u \in D(\varphi H^e) \ \wedge \ u\varphi \in D(\psi H^e)$$

$$\Rightarrow \alpha(u) = e \ \wedge \ u\alpha(\varphi) = u \ \wedge \ \alpha(u\varphi) = e \ \wedge \ u\varphi\alpha(\psi) = u\varphi$$

$$\Rightarrow u\alpha(\varphi\psi) = \alpha(u\varphi\psi)u = \alpha(u\varphi\alpha(\psi))u = \alpha(u\varphi)u = u\alpha(\varphi) = u$$

$$\Rightarrow u \in D((\varphi\psi)H^e);$$

$$u \in D((\varphi\psi)H^e) = D((\varphi H^e)(\psi H^e))$$

$$\Rightarrow ((\varphi\psi)H^e)(u) = u(\varphi\psi) = (u\varphi)\psi = ((\varphi H^e)(u))\psi = ((\varphi H^e)(\psi H^e))(u).$$

Therefore, H^e is a functor from $\underline{K}$ into $\underline{Par}$.

To apply Theorem 2.7, one has to prove the conditions (F$*$), (FI*), and (FM*) as follows:

Ad (F$*$): The function $(H^e)^*\langle X,Y\rangle = d^{Par}_{(X\otimes Y)H^e}(p_1^{X,Y}H^e \times p_2^{X,Y}H^e)$ fulfils

$$((H^e)^*\langle X,Y\rangle)(u) = \left(up_1^{X,Y}, up_2^{X,Y}\right) \in XH^e \times YH^e$$

for each $u \in (X\otimes Y)H^e$, because of $\alpha(p_j^{X,Y}) = 1_{X\otimes Y}$, $j = 1,2$, i.e. $(H^e)^*\langle X,Y\rangle$ is always a total function from $(X\otimes Y)H^e$ into $XH^e \times YH^e$.

Each ordered pair $(u_1,u_2) \in XH^e \times YH^e$ determines uniquely the morphism $d_A(u_1\otimes u_2) \in K[A, X\otimes Y]$. Moreover, this morphism belongs to $(X\otimes Y)H^e$, since $\alpha(d_A(u_1\otimes u_2)) = \alpha(u_1)\alpha(u_2) = ee = e$ ((α19)), and $d_A(u_1\otimes u_2) \in D(d^{Par}_{(X\otimes Y)H^e}(p_1^{X,Y}H^e \times p_2^{X,Y}H^e))$ due to $\alpha(p_j^{X,Y}) = 1_{X\otimes Y}$ and $d_A(u_1\otimes u_2)p_1^{X,Y} = \alpha(u_2)u_1 = \alpha(u_1)u_1 = u_1$, $d_A(u_1\otimes u_2)p_2^{X,Y} = \alpha(u_1)u_2 = \alpha(u_2)u_2 = u_2$, hence $(H^e)^*\langle X,Y\rangle(d_A(u_1\otimes u_2)) = (u_1,u_2)$, i.e. $(H^e)^*\langle X,Y\rangle$ is a surjective function.

The property $d_A(up_1^{X,Y}\otimes up_2^{X,Y}) = ud_{X\otimes Y}(p_1^{X,Y}\otimes p_2^{X,Y}) = u$ shows in conclusion that $(H^e)^*\langle X,Y\rangle$ is an isomorphism in *Par* for all objects $X, Y \in |K|$.

Ad (FI*): The set $IH^e = \{u \in K[A,I] \mid \alpha(u) = e\}$ is the one element set $\{et_A\}$, since

$$\alpha(et_A) = \alpha(e\alpha(t_A)) = \alpha(e) = e$$

and $\alpha(u) = e$ implies

$$et_A = \alpha(u)t_A = d_A(1_A\otimes u)p_1^{A,I}t_A = d_A(t_A\otimes u)(1_I\otimes t_I)r_I$$

$$= d_A(t_A\otimes u)(t_I\otimes 1_I)l_I = d_A(t_A\otimes 1_A)l_Au = u.$$

Therefore, $i_{H^e} : I^{Par} = \{\emptyset\} \to \{et_A\} = IH^e$, defined by $\emptyset \mapsto et_A$, is an isomorphism in *Par* and $i_{H^e} = (t^{Par}_{IH^e})^{-1}$.

Ad (FM*): For every morphism $u \in (X\otimes Y)H^e$ and all morphisms $\varphi \in K[X,U]$, $\psi \in K[Y,V]$ one obtains

$$((H^e)^*\langle X,Y\rangle)(\varphi H^e\times\psi H^e))(u) = (\varphi H^e\times\psi H^e)\Big(\Big(p_1^{X,Y}H^e\Big)(u),\Big(p_2^{X,Y}H^e\Big)(u)\Big)$$

$$= (\varphi H^e\times\psi H^e)\Big(up_1^{X,Y}, up_2^{X,Y}\Big) = \Big(up_1^{X,Y}\varphi, up_2^{X,Y}\psi\Big)$$

and

$$((\varphi\otimes\psi)H^e(H^e)^*\langle U,V\rangle)(u) = (H^e)^*\langle U,V\rangle)(u(\varphi\otimes\psi))$$

$$= \Big(u(\varphi\otimes\psi)p_1^{U,V}, u(\varphi\otimes\psi)p_2^{U,V}\Big) = (u(\varphi\otimes\psi t_V)r_U, u(\varphi t_U\otimes\psi)l_V)$$

$$= (u(\alpha(\varphi)\varphi\otimes\alpha(\psi)t_Y)r_U, u(\alpha(\varphi)t_X\otimes\alpha(\psi)\psi)l_V)$$

$$= (u(\alpha(\varphi)\otimes\alpha(\psi))(1_X\otimes t_Y)r_X\varphi, u(\alpha(\varphi)\otimes\alpha(\psi))(t_X\otimes 1_Y)l_Y\psi)$$

$$= (u\alpha(\varphi\otimes\psi))p_1^{X,Y}\varphi, u\alpha(\varphi\otimes\psi)p_2^{X,Y}\psi)$$

$$= (up_1^{X,Y}\varphi, up_2^{X,Y}\psi),$$

respectively, hence

$$(H^e)^*\langle X,Y\rangle)(\varphi H^e\times\psi H^e) = (\varphi\otimes\psi)H^e(H^e)^*\langle U,V\rangle$$

for all morphisms φ, $\psi\in K$.

The application of Theorem 2.7 shows that $(H^e,((H^e)^*)^{-1},(t_{IH^e}^{Par})^{-1})$ is a d-monoidal functor. ■

Proposition 4.2 (see [3]). *The d-monoidal functor $(H^e,\widetilde{H}^e,i_{H^e}):\underline{K}\to\underline{Par}$ is even dht-monoidal if the conditions*

$$e\neq o_{A,A}\ \textit{and}\ \forall X\in|K|\setminus\{O\}\ (T_K[I,X]=\{\varphi\in K[I,X]\mid \varphi t_X=t_I\}\neq\emptyset)$$

are fulfilled.

Proof. It remains to show the condition (FZ).

Since $e\neq o_{A,A}$, we have $A\neq O$. OH^e is a subset of $K[A,O]=\{o_{A,O}\}$. Because of $\alpha(o_{A,O})=o_{A,A}\neq e$, the set OH^e is the empty set in Par.

Now let $X \neq O$ any object of K. Then

$$T_K[A, X] = \{\varphi \in K[A, X] \mid \varphi t_X = t_A\} \neq \emptyset,$$

since $\exists \psi \in T_K[I, X]$ $(\alpha(t_A\psi) = \alpha(t_A\alpha(\psi)) = \alpha(t_A 1_I) = 1_A)$, hence $t_A\psi \in T_K[A, X]$. Then $et_A\psi$ belongs by $\alpha(et_A\psi) = e$ to XH^e, hence $XH^e \neq \emptyset$. ■

Remark that in every *dhts*-category $\underline{K}$ having the property

$$\forall X \in |K| \setminus \{O\}\ (T_K[I, X] = \{\varphi \in K[I, X] \mid \varphi t_X = t_I\} \neq \emptyset)$$

one obtains by the same reasons as in the proof above

$$\forall X, Y \in |K| \setminus \{O\}\ (T_K[X, Y] \neq \emptyset).$$

Since the set $E_K(A)$ of all subidentities of an object A in a *dhts*-category $\underline{K}$ forms a semilattice with maximal element 1_A and minimal element $o_{A,A}$, there are two particular cases for functors H^e, namely $e = o_{A,A}$ and $e = 1_A$, respectively.

Corollary 4.3. *Let $\underline{K}$ be any dhts-category. Then each object $A \in |K|$ determines the trivial ds-functor $(H^{o_{A,A}}, \widetilde{H^{o_{A,A}}}, i_{H^{o_{A,A}}}) : K \to Par$ defined by*

$$\forall X \in |K|\ (XH^{o_{A,A}} = \{o_{A,X}\}),$$
$$\forall X, Y \in |K| \forall \varphi \in K[X, Y]\ (\varphi H^{o_{A,A}} : \{o_{A,X}\} \to \{o_{A,Y}\},$$
$$o_{A,X} \mapsto o_{A,X}),$$
$$\forall X, Y \in |K|\ (\widetilde{H^{o_{A,A}}}\langle X, Y\rangle : XH^{o_{A,A}} \times YH^{o_{A,A}} \to (X \otimes Y)H^{o_{A,A}},$$
$$(o_{A,X}, o_{A,Y}) \mapsto o_{A,X\otimes Y}),$$
$$i_{H^{o_{A,A}}} : \{\emptyset\} \to IH^{o_{A,A}},\ \emptyset \mapsto o_{A,I}.$$

Proof. If $e = o_{A,A}$, then $XH^e = \{u \in K[A, X] \mid \alpha(u) = e = o_{A,A}\} = \{o_{A,X}\}$ for all $X \in |K|$ since $\alpha(u) = o_{A,A} \Rightarrow u = o_{A,X}$ by $(\alpha 5)$. Therefore, all sets $XH^{o_{A,A}}$ are one element sets and all functions being in consideration are isomorphisms between one element sets. ■

Corollary 4.4. *Let $\underline{K}$ be a dhts-category such that*

$$\forall X \in |K| \setminus \{O\} \ (T_K[I,X] = \{\varphi \in K[I,X] \mid \varphi t_X = t_I\} \neq \emptyset).$$

Then $(H^{1_A}, \widetilde{H^{1_A}}, e_{1_A}) : \underline{K} \to \underline{Par}$ is a dht-monoidal functor for each object $A \in |K| \setminus \{O\}$ which maps every object $X \in |K| \setminus \{O\}$ to the set $T_K[A,X]$ of all total morphisms $u \in K[A,X]$, i.e. the usual Hom-functor $H^A : T_K \to Set$ is a restriction of $H^{1_A} : K \to Par$.

Proof. It remains to show that $\varphi H^{1_A} : T_K[A,X] \to T_K[A,Y]$ is a total function for all $\varphi \in T_K[X,Y]$, X, $Y \in |K| \setminus \{O\}$. Because of ($\alpha$4), one has $\alpha(\varphi) = 1_X$ for $\varphi \in T_K[X,Y]$, hence $D(\varphi H^{1_A}) = \{u \in T_K[A,X] \mid u\alpha(\varphi) = u1_X = u\} = T_K[X,Y]$, thus φH^{1_A} id a total function. ∎

The functors H^e related to subidentities e in *dhts*-categories represent an important tool for the construction of full, faithful, and representative functors from a *dhts*-category $\underline{K}$ into $\underline{Par}$, see the papers by J. Schreckenberger [3] and [4].

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