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COMPLETION OF A HALF LINEARLY CYCLICALLY ORDERED GROUP

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Abstract

The notion of a half *lc*-group G is a generalization of the notion of a half linearly ordered group. A completion of G by means of Dedekind cuts in linearly ordered sets and applying Świerczkowski's representation theorem of *lc*-groups is constructed and studied.

Keywords: dedekind cut, cyclically ordered group, *lc*-group, half *lc*-group, completion of a half *lc*-group.

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J. Jakubík [3] introduced and studied the notions of a half cyclically ordered group and a half linearly cyclically ordered group (half *lc*-group) as a generalization of the notions of a half partially ordered group and a half linearly ordered group that were introduced by M. Giraudet and F. Lucas [2].

A completion $C(H)$ of a linearly cyclically ordered set H has been defined and studied by V. Novák [5] and by V. Novák and M. Novotný [6].

In [4] it is defined and investigated a completion H^* of a linearly cyclically ordered group H. A new construction of a completion $M(H)$ of H by using the Dedekind cuts method in linearly ordered sets is contained in [1]. It is proved that $M(H) = H^*$.

Half *lc*-groups are dealt with in Section 4. If a half *lc*-group is at the same time a half linearly ordered group, then its decreasing part consists of elements of the second order ([2], Proposition I.2.2). The question of the existence of such elements in an arbitrary half lc-group, is open.

Let G be a half lc-group with the increasing part H. In this paper there is presented a completion $M_h(G)$ of G. It is shown that $M(H)$ is the increasing part of a half lc-group $M_h(G)$. G is called *M_h-complete* if $M_h(G) = G$. Necessary and sufficient conditions are found for G to be M_h-complete (Theorem 4.12).

1. Preliminaries

Assume that A and B are linearly ordered sets. Let $S = \{(a, b) : a \in A, b \in B\}$. We put $(a_1, b_1) \le (a_2, b_2)$ whenever $b_1 < b_2$ or $b_1 = b_2$ and $a_1 \le a_2$ for each $(a_1, b_1), (a_2, b_2) \in S$. Then the linearly ordered set S is said to be the *lexicographic product* of A and B and the notation $S = A \circ B$ will be used.

Let L be a linearly ordered set and let X be a subset of L. Denote by $X^u(X^l)$ the set of all upper (lower) bounds of X in L. Futher we denote by $D(L)$ the system of all subsets of L of the form $(X^u)^l$ where X is a nonempty and upper bounded subset of L. Elements of $D(L)$ are called *Dedekind cuts* of L. If the system $D(L)$ is partially ordered by inclusion, then $D(L)$ is a linearly ordered set. The mapping $\varphi(x) = (\{x\}^u)^l$ is an isomorphism of the linearly ordered set L into $D(L)$. In the next the elements x and $\varphi(x)$ will be identified. Then L is a subset of $D(L)$ and the following conditions are fulfilled:

(α_1) For each element $c \in D(L)$ there exist nonempty subsets X and Y of L such that X is upper bounded, Y is lower bounded in L and $c = \sup(X) = \inf(Y)$ in $D(L)$.

(α_2) For each nonempty and upper (lower) bounded subset X (Y) of L there exists an element $c \in D(L)$, $c = \sup(X)$ ($c = \inf(Y)$) in $D(L)$.

If A is a subset of L and $a = \sup(A)$ ($a = \inf(A)$) in L, then $a = \sup(A)(a = \inf(A))$ in $D(L)$.

For the following two definitions cf. Novák [5].

Definition 1.1. Let M be a nonempty set and let C be a ternary relation on M with the following properties:

(I) If $(x, y, z) \in C$, then $(z, y, x) \notin C$.

(II) If $(x, y, z) \in C$, then $(y, z, x) \in C$.

(III) If $(x, y, z) \in C, (x, z, u) \in C$, then $(x, y, u) \in C$.

Then C is said to be a *cyclic order* on M and the pair $(M; C)$ is called a *cyclically ordered set.*

If A is a subset of M, then A is considered as being cyclically ordered by the inherited cyclic order.

Definition 1.2. Let $(M; C)$ be a cyclically ordered set satisfying the following condition:

(IV) If x, y and z are distinct elements of M, then either $(x, y, z) \in C$ or $(z, y, x) \in C$.

Then C is said to be an *l-cyclic order* on M and $(M; C)$ is called an *l-cyclically ordered set.*

Several terms are used in papers for the term "l-cyclic order". Namely, "l-cyclic order" is called "cyclic order" in [8], "complete cyclic order" in [7], and "linear cyclic order in [5]".

Definition 1.3 (cf. Rieger [8]). Let $(H; +)$ be a group and let $(H; C)$ be a cyclically ordered set satisfying the condition

(V) if $x, y, z, a, b \in H$ such that $(x, y, z) \in C$, then $(a + x + b, a + y + b, a + z + b) \in C$.

Then $(H; +, C)$ is called a *cyclically ordered group.* If C is an l-cyclic order, then $(H; +, C)$ is called an *lc-group.*

Each subgroup of a cyclically ordered group is a cyclically ordered group.

Example 1.4. Let $(L; +, \leq)$ be a linearly ordered group and let $x, y, z \in L$. We put

(g) $(x, y, z) \in C_1$ if and only if $x < y < z$ or $y < z < x$ or $z < x < y$.

Then $(L; +, C_1)$ is an lc-group. We say that the l-cyclic order C_1 is *generated by the linear order* $\leq$ *on* L.

In the next, if $(S; \leq)$ is a linearly ordered set then S is assumed to be l-cyclically ordered with the l-cyclic order defined by (g).

Example 1.5. Let K be the set of all reals k such that $0 \leq k < 1$ with the natural linear order. Denote by C_2 the l-cyclic order on K defined by (g). The group operation $+$ on K is defined as addition mod 1. Then $(K; +, C_2)$ is an lc-group.

We want to define a ternary relation C on the direct product $L \times K$ of the groups L and K. Let $u = (x, k_1), v = (y, k_2), w = (z, k_3) \in L \times K$. We put $(u, v, w) \in C$ if and only if some of the following conditions is satisfied:

(i) $(k_1, k_2, k_3) \in C_2$,

(ii) $k_1 = k_2 \neq k_3$ and $x < y$,

(iii) $k_2 = k_3 \neq k_1$ and $y < z$,

(iv) $k_3 = k_1 \neq k_2$ and $z < x$,

(v) $k_1 = k_2 = k_3$ and $(x, y, z) \in C_1$.

Then $(L \times K; C)$ is an lc-group which is denoted by $L \otimes K$.

An isomorphism of cyclically ordered groups is defined in a natural way.

Theorem 1.6 (Świerczkowski [9]). *Let H be an lc-group. Then there exists a linearly ordered group L such that H is isomorphic to a subgroup of $L \otimes K$.*

In the next H will be considered as a subgroup of $L \otimes K$. We denote

$$L_1 = \{x \in L : \text{ there exists } k \in K \text{ with } (x, k) \in H\},$$

$$K_1 = \{k \in K : \text{ there exists } x \in L \text{ with } (x, k) \in H\},$$

$$H_0 = \{h \in H : \text{ there exists } x \in L \text{ with } h = (x, 0)\}.$$

Then L_1 is a subgroup of L, K_1 is a subgroup of K, and H_0 is an invariant subgroup of H. Moreover, H_0 is a linearly ordered group if we put $h > 0$ if and only if $x > 0$. It can happen that $H_0 = \{0\}$.

Let $(G; +)$ be a group and let $(G; C)$ be a cyclically ordered set, $x, y, z \in G$. Form the sets

$$G\uparrow = \{g \in G : (x, y, z) \in C \Longrightarrow (g + x, g + y, g + z) \in C\},$$

$$G\downarrow = \{g \in G : (x, y, z) \in C \Longrightarrow (g + z, g + y, g + x) \in C\}.$$

Definition 1.7. (cf. Jakubík [3].) Let $(G; +)$ be a group and let $(G; C)$ be a cyclically ordered set such that the following conditions are fulfilled:

(1) the system C is nonempty;

(2) if $g \in G$ and $(x, y, z) \in C$, then $(x + g, y + g, z + g) \in C$;

(3) $G = G \uparrow \cup G \downarrow$;

(4) if $(x, y, z) \in C$, then either $\{x, y, z\} \subseteq G \uparrow$ or $\{x, y, z\} \subseteq G \downarrow$.

Then $(G; +, C)$ is said to be a *half cyclically ordered group.*

$G \uparrow$ ($G \downarrow$) is called the *increasing* (*decreasing,* resp.) part of G.

If $(G; +, C)$ is a half cyclically ordered group, then $G \uparrow$ is a cyclically ordered group. If $G \uparrow$ is an lc-group, then $(G; +, C)$ is called a *half lc-group.*

Let $(G; +, C)$ be a half cyclically ordered group and let G' be a subgroup of G with the nonempty inherited cyclic order C'. Then $(G; +, C')$ is called an *hc-subgroup* of $(G; +, C)$.

We shall often write briefly G instead of $(G; +, C)$ or $(G; C)$.

Each cyclically ordered group with a nontrivial cyclic order is a half cyclically ordered group with $G \uparrow = G$ and $G \downarrow = \emptyset$.

If $x, y \in G \uparrow$ and $u, v \in G \downarrow$, then $x + y \in G \uparrow$, $u + v \in G \uparrow$, $x + u \in G \downarrow$, $u + x \in G \downarrow$. This follows from 1.7.

2. COMPLETION OF AN L-CYCLICALLY ORDERED SET

The definitions and results in this section are due to Novák [5].

Assume that (H, C) is an l-cyclically ordered set and let $x \in H$. If, for each $y, z \in H$, we put $y <_x z$ if and only if either $(x, y, z) \in C$ or $x = y \neq z$, then $<_x$ is a linear order on H with the least element x.

Definition 2.1. A linear order $<$ on H is called a *cut* on $(H; C)$ if the cyclic order generated by the linear order $<$ coincides with the original cyclic order C on H.

The linear order $<_x$ is a cut on $(H; C)$.

Let $<$ be a cut on $(H; C)$. The following three cases can occur:

(i) $(H; <)$ has the least and the greatest element.

(ii) $(H; <)$ has neither the least nor the greatest element.

(iii) $(H; <)$ has either the least or the greatest element.

In the case (ii) a cut $<$ is called a *gap.* If $(H; C)$ contains no gaps, then it is called *complete.*

Definition 2.2. A cut $<$ on $(H; C)$ is said to be *regular* if some of the following conditions is satisfied:

(i) $<$ is a gap,

(ii) $(H; <)$ has the least element.

Denote by $\mathcal{R}(H)$ the set of all regular cuts on $(H; C)$. Let $c_1 = <_1$, $c_2 = <_2$, and $c_3 = <_3$ be distinct elements of $\mathcal{R}(H)$. We put $(c_1, c_2, c_3) \in \bar{C}$ if and only if there are elements $x, y, z \in H$ such that

$$x <_1 y <_1 z, \ y <_2 z <_2 x, \ z <_3 x <_3 y.$$

For each $x \in H$ we put $\varphi(x) = <_x$.

Theorem 2.3 (cf. [5], 4.2 and 4.3). *$(\mathcal{R}(H); \bar{C})$ is an l-cyclically ordered set and φ is an isomorphism of the l-cyclically ordered set H into $\mathcal{R}(H)$.*

Elements x and $\varphi(x)$ will be identified. Hence H is considered as a subset of $\mathcal{R}(H)$. $\mathcal{R}(H)$ is a complete l-cyclically ordered set and it is said to be a *completion* of H.

3. Completion of an lc-group

In the whole section H is assumed to be an lc-group. A construction of a completion $M(H)$ of H will be recalled (cf. [1]) and some auxiliary results will be derived.

Let $L_1, K_1, L_1 \otimes K_1$ be as in Section 1. The linear order on the lexicographic product $L_1 \circ K_1$ of the linearly ordered sets L_1 and K_1 is a cut on the l-cyclically ordered set $L_1 \otimes K_1$ and H is a subset of $L_1 \circ K_1$.

Therefore, $D(H)$ can be considered as a subset of $D(L_1 \circ K_1)$. We have $H \subseteq D(H) \subseteq D(L_1 \circ K_1)$.

If the system $\bar{D}(H) = D(H) \cup \{H\}$ is partially ordered by inclusion, then $\{H\}$ is the greatest element of the chain $\bar{D}(H)$.

Lemma 3.1 (cf. [1], 3.4). *The l-cyclically ordered set $\bar{D}(H)$ is isomorphic to $\mathcal{R}(H)$. $\mathcal{R}(H)$ and $\bar{D}(H)$ will be identified.*

Let $c \in \bar{D}(H),\ A \subseteq H,\ k \in K_1$. Denote

$$A_k = \{a \in A:\ a = (x,k) \text{ for some } x \in L_1\},$$

$$A(L_1) = \{x \in L_1\text{: there exists } k_1 \in K_1 \text{ with } (x,k_1) \in A\},$$

$$A(K_1) = \{k_1 \in K_1\text{: there exists } x \in L, \text{ with } (x,k_1) \in A\},$$

$$U(c) = \{u \in H:\ u \geq c\},\ V(c) = \{v \in H:\ v \leq c\}.$$

Then according to (α_1) we obtain

$$c = \sup(V(c)) = \inf(U(c)) \textit{ in } \bar{D}(H).$$

Let $c_1, c_2 \in \bar{D}(H)$. Then

$$c_1 = \sup(V(c_1)) = \inf(U(c_1)), \text{ and } c_2 = \sup(V(c_2)) = \inf(U(c_2)) \text{ in } \bar{D}(H).$$

Now, we intend to define the operation $+$ on $\bar{D}(H)$.

If for all elements $v_1 = (x,k_1) \in V(c_1), v_2 = (y,k_2) \in V(c_2)$ the relation $k_1 +_r k_2 < 1$ holds, where $+_r$ is the usual operation on the group of reals, then we put

$$c_1 + c_2 = \sup\{v_1 + v_2:\ v_1 \in V(c_1), v_2 \in V(c_2)\} \text{ in } \bar{D}(H).$$

If there are elements $v_1 \in V(c_1), v_2 \in V(c_2)$ such that $k_1 +_r k_2 \geq 1$, then we put

$$c_1 + c_2 = \sup\{v_1 + v_2:\ v_1 \in V(c_1), v_2 \in V(c_2), k_1 +_r k_2 \geq 1\} \text{ in } \bar{D}(H).$$

Then $(\bar{D}(H); +)$ is a semigroup and $0 \in H$ is a neutral element of $(\bar{D}(H); +)$. If $M(H)$ is the set of all elements of $\bar{D}(H)$ that have an inverse in $\bar{D}(H)$, then $M(H)$ is a group. The *lc*-group $M(H)$ (with the inherited cyclic order from $\bar{D}(H)$) is said to be a *completion* of H. $M(H)$ is a maximal subsemigroup of $\bar{D}(H)$ being a group and H is a subgroup of $M(H)$.

Remark that the notion of a completion H^* of H was defined also in [4] in a formally different way. It was proved in [1] that $M(H) = H^*$.

If $M(H) = H$, then H is called *M-complete.* From the definition of H^*, it follows that $(H^*)^* = H^*$. Therefore, $M(H)$ is M-complete.

At first $M(H)$ will be investigated under the assumption $H_0 \neq \{0\}$ and then under that $H_0 = \{0\}$.

Suppose that $H_0 \neq \{0\}$.

Let $c \in \bar{D}(H)$. Assume that the set $V(c)(K_1)$ has the greatest element $k \in K_1$ which is at the same time the least element of $U(c)(K_1)$. Then we say that c is of type (τ). Therefore, the sets $(V(c))_k$ and $(U(c))_k$ are nonempty and we have

$$c = \sup(V(c))_k = \inf(U(c))_k \text{ in } \bar{D}(H). \tag{1}$$

Let $c, c_i \in \bar{D}(H)(i = 1, 2, 3)$ be elements of type (τ). If no misunderstanding can occur, the corresponding greatest elements of $V(c)(K_1)$ and $V(c_i)(K_1)$ will be denoted by k and k_i $(i = 1, 2, 3)$, respectively.

By (1), we have

$$c_1 = \sup\left(V(c_1)\right)_{k_1}, \text{ and } c_2 = \sup\left(V(c_2)\right)_{k_2} \text{ in } \bar{D}(H).$$

The definition of the operation $+$ in $\bar{D}(H)$ implies

$$c_1 + c_2 = \sup\{v_1 + v_2 : \ v_1 \in \left(V(c_1)\right)_{k_1}, \ v_2 \in \left(V(c_2)\right)_{k_2}\} \text{ in } \bar{D}(H). \tag{2}$$

Evidently, that $c_1 + c_2$ is of type (τ).

Let $c \in \bar{D}(H)$ be of type (τ), $S, T \subseteq H$, and $c = \sup(S) = \inf(T)$ in $\bar{D}(H)$. Then k is the greatest element of $S(K)$ and the least element of $T(K)$. Therefore, S_k and T_k are nonempty subsets of H and we have

$$c = \sup(S_k) = \inf(T_k) \text{ in } \bar{D}(H). \tag{3}$$

Let $w_1, w_2 \in H$, $w_1 = (x_1, k)$, and $w_2 = (x_2, k)$. Evidently, $w_1 \leq w_2$ implies $w_1 + w \leq w_2 + w$ and $w + w_1 \leq w + w_2$ for each $w \in H$. This result will be applied in the sequel.

Lemma 3.2. *Let c_1, c_2 be elements of $\bar{D}(H)$ of type (τ), $S_1, S_2 \subseteq H$, and let $c_1 = \sup S_1, c_2 = \sup S_2$ in $\bar{D}(H)$. Then*

$$c_1 + c_2 = \sup\{s_1 + s_2 : \ s_1 \in S_{1k_1}, \ s_2 \in S_{2k_2}\} \textit{ in } \bar{D}(H).$$

Proof. There exists $c \in \bar{D}(H)$, $c = \sup\{s_1 + s_2 :\ s_1 \in S_{1k_1},\ s_2 \in S_{2k_2}\}$. Therefore, c is of type (τ), $k = k_1+k_2$ is the greatest element of $V(c)(K_1)$ and also the least element of $U(c)(K_1)$. From $S_{1k_1} \subseteq (V(c_1))_{k_1}, S_{2k_2} \subseteq V(c_2)_{k_2}$ and from (2), we infer that $c \le c_1+c_2$. We are going to show that $c_1+c_2 \le c$, i.e., $(U(c))_k \subseteq (U(c_1+c_2))_k$. Let $h \in (U(c))_k$. Then $h \ge s_1 + s_2$ for each $s_1 \in S_{1k_1},\ s_2 \in S_{2k_2}, -s_1+h \ge s_2$ for each $s_2 \in S_{2k_2}$. With respect to (3) and (1), we get $-s_1 + h \ge c_2 \ge v_2$ for each $v_2 \in (V(c_2))_{k_2}$. By using (3) and (1), from $h - v_2 \ge s_1$ for each $s_1 \in S_{1k_1}$, it follows that $h - v_2 \ge c_1 \ge v_1$ for each $v_1 \in (V(c_1))_{k_1}$, and so $h \ge v_1 + v_2$ for each $v_1 \in (V(c_1))_{k_1}, v_2 \in (V(c_2))_{k_2}$. In view of (2), we get $h \ge c_1 + c_2$. We conclude that $h \in (U(c_1 + c_2))_k$. ■

Lemma 3.3 (cf. [1], 3.6 and 3.9). *Let $c \in \bar{D}(H)$.*

(i) *If $c = \{H\}$, then $c \notin M(H)$.*

(ii) *If $c \neq \{H\}$, then $c \in M(H)$ if and only if the following conditions are satisfied in H:*

$$(\mathrm{p}_1)\qquad \begin{aligned} &\inf\{u - v :\ u \in U(c), v \in V(c)\} = 0,\\ &\inf\{-v + u :\ u \in U(c), v \in V(c)\} = 0. \end{aligned}$$

(iii) If $c \in M(H)$, then c is of type (τ).

Lemma 3.4. *Let $c \in \bar{D}(H)$ be of type (τ), $S, T \subseteq H$, $c = \sup(S) = \inf(T)$ in $\bar{D}(H)$. Then $c \in M(H)$ if and only if the following conditions are satisfied in H:*

(p_2) $\inf\{t - s :\ s \in S_k, t \in T_k\} = 0$ and $\inf\{-s + t :\ s \in S_k, t \in T_k\} = 0$.

Proof. Let $c \in \bar{D}(H)$ be of type (τ). Hence $c \neq \{H\}$. In view of Lemma 3.3, we prove that the conditions (p_1) and (p_2) are equivalent. It suffices to show that (p_1) implies (p_2).

Assume that (p_1) holds. With respect to (3), we get $c = \sup(S_k) = \inf(T_k)$ in $\bar{D}(H)$. From $s \le t$, we infer $t - s \ge 0$ for each $s \in S_k,\ t \in T_k$.

Assume that $d \in H$, $d = (x, k')$, $d \leq t - s$ for each $s \in S_k$, $t \in T_k$. Hence $k' = 0$. We have to prove that $d \leq 0$. Since $d + s \leq t$ for each $t \in T_k$, $d + s \leq c$. Therefore, $d + s \leq u$ for each $u \in (U(c))_k$, and $s \leq -d + u$ for each $s \in S_k$. This implies that $c \leq -d + u$ and so $v \leq -d + u$ for each $v \in (V(c))_k$. Hence $d \leq u - v$ for each $u \in (U(c))_k$, $v \in (V(c))_k$ and then also for each $u \in U(c)$, $v \in V(c)$. The condition (p_1) implies $d \leq 0$. The remaining case is similar. ∎

The following lemma is easy to verify.

Lemma 3.5. *Let c_1, c_2 and c be elements of $\bar{D}(H)$ of type (τ) such that $k_1 = k_2$. If $c_1 \leq c_2$, then $c_1 + c \leq c_2 + c$ and $c + c_1 \leq c + c_2$.*

Lemma 3.6. *Let $c_1, c_2 \in M(H), S_i, T_i \subseteq H, c_i = \sup(S_i) = \inf(T_i)$ $(i = 1, 2)$ in $\bar{D}(H)$. Then*

$$c_1 + c_2 = \inf\{t_1 + t_2 : \ t_1 \in T_{1k_1}, t_2 \in T_{2k_2}\} \ \text{ in } \ \bar{D}(H).$$

Proof. Let $c_1, c_2 \in M(H)$. According to Lemma 3.3, c_1 and c_2 are of type (τ). We have $s_1 + s_2 \leq t_1 + t_2$ for each $s_i \in S_{ik_i}, t_i \in T_{ik_i} (i = 1, 2)$. Denote $c = c_1 + c_2$ and $c' = \inf\{t_1 + t_2 : \ t_1 \in T_{1k_1}, \ t_2 \in T_{2k_2}\}$. Since $c \in M(H), c$ is of type (τ). For the greatest element k of $(V(c))(K_1)$, we have $k = k_1 + k_2$. The element c' is also of type (τ) and k is the greatest element of $(V(c'))(K_1)$. With respect to Lemma 3.2, we have $c = \sup\{s_1 + s_2 : \ s_1 \in S_{1k_1}, s_2 \in S_{2k_2}\}$. Then $c \leq c'$. We have to show that $c' \leq c$, i.e., $(V(c'))_k \subseteq (V(c))_k$. Let $h \in (V(c'))_k$. From $h \leq c'$, we infer $h \leq t_1 + t_2$ for each $t_1 \in T_{1k_1}, t_2 \in T_{2k_2}$. Hence $h - t_2 \leq t_1$ for each $t_1 \in T_{1k_1}$ and so $h - t_2 \leq c_1$. Applying Lemma 3.5 and $c_1 \in M(H)$, we get $-c_1 + h \leq t_2$ for each $t_2 \in T_2$. This yields $-c_1 + h \leq c_2$. Again by using Lemma 3.5 and $c_1 \in M(H)$, we obtain $h \leq c$. Therefore, $h \in (V(c))_k$. ∎

By summarising the previous results, we get:

Theorem 3.7. *Let $H_0 \neq \{0\}$. The lc-group $M(H)$ has the following properties:*

(a) *$M(H)$ is M-complete;*

(b) *H is a subgroup of $M(H)$;*

(c) *for each element* $c \in M(H)$ *there exist* $k \in K_1$ *and* $S, T \subseteq H$ *such that* S_k *and* T_k *are nonempty subsets of* H, *and* $c = \sup(S_k) = \inf(T_k)$ *in* $M(H)$. ∎

Theorem 3.8. *Let* $H_0 \neq \{0\}$. *Assume that* H' *is an lc-group fulfilling the conditions* (a)–(c) (*with* H' *instead of* $M(H)$). *Then there exists an isomorphism* ϕ *of the lc-group* $M(H)$ *onto* H'*such that* $\phi(h) = h$ *for each* $h \in H$.

Proof. Assume that $c \in M(H)$. According respect to (c), there exist $k \in K_1, S, T \subseteq H$ such that $c = \sup(S_k) = \inf(T_k)$ in $M(H)$ (recall that k is the greatest (least) element of $S(K_1)(T(K_1)))$. Let $Z_1 = \{t - s : \ t \in T_k, s \in S_k\}$ and $Z_2 = \{-s + t : \ s \in S_k, t \in T_k\}$. With respect to Lemma 3.4, we get $\inf(Z_1) = \inf(Z_2) = 0$ in H. Let $T' = \{h' \in H' : \ h' \geq s$ for each $s \in S_k\}$ and $S' = \{h' \in H' : \ h' \leq t'$ for each $t' \in T'\}$. There exists $c' \in D(H')$ with $c' = \sup(S') = \inf(T')$ in $D(H')$. We have $c' = \sup(S'_k) = \inf(T'_k)$ in $D(H')$. Let us denote $Z'_1 = \{t' - s' : \ s' \in S'_k, t' \in T'_k\}$ and $Z'_2 = \{-s' + t' : \ s' \in S'_k, \ t' \in T'_k\}$. We get $\inf(Z_1) = \inf(Z'_1) = 0, \inf(Z_2) = \inf(Z'_2) = 0$ in H'. Then Lemma 3.4 yields that $c' \in M(H')$. According to (a), $M(H') = H'$ and so $c' \in H'$.

We put $\phi(c) = c'$. It is easy to verify that ϕ is correctly defined and that ϕ is an isomorphism of the lc-group $M(H)$ onto H' with $\phi(h) = h$ for each $h \in H$. ∎

Now assume that $H_0 = \{0\}$. We may suppose that H is a subgroup of K. If H is finite then $M(H) = H$. If H is infinite, then the lc-group $M(H)$ is isomorphic to K (cf. [4] and [1]).

In both cases $H_0 \neq \{0\}$ and $H_0 = \{0\}$ the following theorem holds.

Theorem 3.9 (cf. [4], 7.5). *Let* H *be an lc-group. Then* H *is* M*-complete if and only if some of the following conditions is satisfied:*

(i) H *is finite;*

(ii) H *isomorphic to* K;

(iii) $H_0 \neq \{0\}$ *and* H_0 *is* M*-complete.*

4. Completion of a half *lc*-group

In the present section we suppose that G is a half *lc*-group with a cyclic order C and with $G\downarrow \neq \emptyset$. Then G fails to be an *lc*-group.

We shall use the notations $G\uparrow = H$ and $G\downarrow = H'$. As in the previous sections $H \subseteq L_1 \circ K_1$ and $D(H) \subseteq D(L_1 \circ K_1)$. Assume that there exists an element $a \in H'$ of the second order. The mapping $\psi : H \to H'$ defined by $\psi(h) = a + h$ is a bijection reversing the *l*-cyclic order of H. If for each $h_1, h_2 \in H$ we set $a + h_1 \leq a + h_2$ if and only if $h_2 \leq h_1$, then $a + H$ is a linearly ordered set. We have $h_1 + a \leq h_2 + a$ if and only if $h_1 \leq h_2$.

Assume that $H_0 \neq \{0\}$.

Lemma 4.1 (cf. [3], 3.6). *H_0 is a normal subgroup of G.*

Lemma 4.2 (cf. [3], 3.8). *$A = H_0 \cup (a + H_o)$ is a half lc-subgroup of G. Moreover, A is a half linearly ordered group.*

Lemma 4.3. *Let $h_1,\ h_2 \in H, h_1 = (x_1, k_1),\ h_2 = (x_2, k_2),\ a + h_1 + a = (x_1', k_1')$, and $a + h_2 + a = (x_2', k_2')$. Then $k_1 = k_2$ if and only if $k_1' = k_2'$.*

Proof. Let $k_1 = k_2$. Then $h_1 - h_2 \in H_0$. Using Lemma 4.1 we get $a + h_1 + a - (a + h_2 + a) = a + (h_1 - h_2) + a \in H_0$. Hence $k_1' = k_2'$. The converse is analogous. ■

Lemma 4.4. *Let $h_1, h_2 \in H,\ h_1 = (x_1, k)$ and $h_2 = (x_2, k)$. Assume that $h_1 < h_2$. Then $a + h_2 + a < a + h_1 + a$.*

Proof. Let $a + h_1 + a = (x, k_1)$ and $a + h_2 + a = (y, k_2)$. By Lemma 4.3, we get $k_1 = k_2$.

If $k = 0$, then $h_1, h_2 \in H_0$ and the assertion follows from Lemma 4.2.

If $k \neq 0$, then $k_1 \neq 0$ as well and $0 < h_1 < h_2$ yields that $(0, h_1, h_2) \in C$. This implies that $(a + h_2 + a, a + h_1 + a, 0) \in C$. Hence $y < x$ and thus $a + h_2 + a < a + h_1 + a$. ■

Assume that $c_1, c_2 \in \bar{D}(H)$ are of type $(\tau), S_i, T_i \subseteq H, c_i = \sup(S_i) = \inf(T_i) (i = 1, 2)$ in $\bar{D}(H)$ and that $k_i \in K_1$ corresponds to c_i $(i = 1, 2)$ as in Section 3. Then $c_i = \sup(S_{ik_i}) = \inf(T_{ik_i})$ $(i = 1, 2)$ in $\bar{D}(H)$.

Let $s_i \in S_{ik_i}, t_i \in T_{ik_i} (i = 1, 2)$. From $s_1 \leq t_1,\ s_2 \leq t_2$ for each $s_i \in S_{ik_i}, t_i \in T_{ik_i} (i = 1, 2)$, we obtain $a + t_1 + a + s_2 \leq a + t_1 + a + t_2$. According to Lemma 4.4 we get $a + t_1 + a + t_2 \leq a + s_1 + a + t_2$. Hence $a + t_1 + a + s_2 \leq a + s_1 + a + t_2$. Thus there exist $\sup\{a + t_1 + a + s_2 :\ s_2 \in S_{2k_2}, t_1 \in T_{1k_1}\}$ and $\inf\{a + s_1 + a + t_2 :\ s_1 \in S_{1k_1}, t_2 \in T_{2k_2}\}$ in $\bar{D}(H)$.

Lemma 4.5. *Let $S_i, T_i \subsetneqq H, c_i \in M(H), c_i = \sup(S_i) = \inf(T_i)$ $(i = 1, 2), c \in \bar{D}(H)$, and $c = \sup\{a+t_1+a+s_2 :\ s_2 \in S_{2k_2}, t_1 \in T_{1k_1}\}$ in $\bar{D}(H)$. Then*

(i) *$c \in M(H)$,*

(ii) *$c = \inf\{a + s_1 + a + t_2 :\ s_1 \in S_{1k_1}, t_{2k_2}\}$ in $\bar{D}(H)$.*

Proof. (i) We have to prove that there exists an inverse to c in $\bar{D}(H)$. By Lemma 3.3 elements c_1 and c_2 are of type (τ). Hence c is of type (τ) as well. Denote $B = \{a + t_1 + a + s_2 :\ s_2 \in S_{2k_2}, t_1 \in T_{1k_1}\}, D = \{a + s_1 + a + t_2 :\ s_1 \in S_{1k_1}, t_2 \in T_{2k_2}\}$. For the element $k \in K_1$ corresponding to c we have $k = k_1 + k_2$, k is the greatest element of $B(K_1)$ and the least element of $D(K_1)$. From $b \leq d$ for each $b \in B, d \in D, b = (x, k), d = (y, k)$, we infer that $d - b \geq 0$. Let $h \in H, h \leq d - b$ for each $b \in B, d \in D$. Then $h \in H_0, h = (z, 0)$. We have $h \leq a+s_1+a+t_2-(a+t_1+a+s_2) = a+s_1+a+t_2-s_2+a-t_1+a \in H_0$. This yields that $a-s_1+a+h+a+t_1+a \leq t_2-s_2$ for each $s_2 \in S_{2k_2}, t_2 \in T_{2k_2}$. Since $c_2 \in M(H)$, by using Lemma 3.4, we obtain $\inf\{t_2 - s_2 :\ s_2 \in S_{2k_2}, t_2 \in T_{2k_2}\} = 0$ in H. Then $a-s_1+a+h+a+t_1+a \leq 0,\ a+h+a \geq s_1 - t_1,\ a-h+a \leq t_1 - s_1$ for each $s_1 \in S_{1k_1}, t_1 \in T_{1k_1}$. Since $c_1 \in M(H)$, Lemma 3.4 implies $a - h + a \leq 0, h \leq 0$.
Therefore

$$(*) \qquad \inf\{d - b :\ b \in B, d \in D\} = 0 \text{ in } H.$$

In an analogous way, we get $\inf\{-b + d \colon b \in B, d \in D\} = 0$ in H.

We have $-d \leq -b$ for each $b \in B, d \in D$. Hence the set $-D = \{-d \in H\colon d \in D\}$ is nonempty and upper bounded. Hence there exists $c' \in \bar{D}(H), c' = \sup(-D)$. We have $c + c' = \sup\{b + d\colon b \in B, d \in -D\} = \sup\{b - d\colon b \in B, d \in D\} = \inf\{d - b\colon b \in B, d \in D\}$ in $\bar{D}(H)$. By using $(*)$, we get $\inf\{d - b\colon b \in B, d \in D\} = 0$ in $\bar{D}(H)$. Thus $c + c' = 0$. Analogously, we get $c' + c = 0$. We conclude that c' is an inverse to c in $\bar{D}(H)$.

(ii) The proof is analogous to that of Lemma 3.6. ■

We denote

$$a + M(H) = \{a + c\colon c \in M(H)\},$$

$$M_h(G) = M(H) \cup (a + M(H)).$$

Recall that $\bar{D}(H)$ and $\mathcal{R}(H)$ are identified. The l-cyclic order on $M(H) \subseteq \bar{D}(H)$ is denoted by the same symbol $\bar{C}$ as on $\mathcal{R}(H)$.

Let $c_1, c_2, c_3 \in M(H)$. We define the ternary relation $\bar{C}_1$ on $M_h(G)$ to coincide with $\bar{C}$ on $M(H)$ and with C on G. Further we put $(a+c_3, a+c_2, a+c_1) \in \bar{C}_1$ if and only if $(c_1, c_2, c_3) \in \bar{C}$. If $\bar{a}, \bar{b}, \bar{c} \in M_h(G), (\bar{a}, \bar{b}, \bar{c}) \in \bar{C}_1$, then either $\{\bar{a}, \bar{b}, \bar{c}\} \subseteq M(H)$ or $\{\bar{a}, \bar{b}, \bar{c}\} \subseteq a + M(H)$. Therefore, $M_h(G)$ is a cyclically ordered set.

We intend to define a binary operation $+$ on $M_h(G)$ to coincide with the group operations $+$ on $M(H)$ and G.

Let $c_i \in M(H), S_i, T_i \subseteq H, c_i = \sup(S_i) = \inf(T_i)$ $(i = 1, 2)$.
Then $c_i = \sup(S_{ik_i}) = \inf(T_{ik_i})$ $(i = 1, 2)$ in $\bar{D}(H)$.

As before, we put

$$c_1 + c_2 = \sup\{s_1 + s_2 :\ s_1 \in S_{1k_1},\ s_2 \in S_{2k_2}\} \text{ in } \bar{D}(H).$$

Further we put

$$(a + c_1) + (a + c_2) = \sup\{a + t_1 + a + s_2 :\ s_2 \in S_{2k_2}, t_1 \in T_{1k_1}\} \text{ in } \bar{D}(H),$$

$$c_1 + (a + c_2) = a + ((a + c_1) + (a + c_2)),$$

$$(a + c_1) + c_2 = a + (c_1 + c_2).$$

According to Lemma 4.5, we have $(a + c_1) + (a + c_2) \in M(H)$.

Lemma 4.6. $(M_h(G), +)$ *is a group.*

Proof. We begin with the proof that $+$ is an associative operation on $M_h(G)$.

Denote $(a + c_1) + (a + c_2) = c$ and $(a + c_2) + (a + c_3) = c'$. Hence $c' = \sup\{a + t_2 + a + s_3 :\ s_3 \in S_{3k_3}, t_2 \in T_{2k_2}\}$. In view of Lemma 4.5, we have $c = \inf\{a + s_1 + a + t_2 :\ s_1 \in S_{1k_1}, t_2 \in T_{2k_2}\}$.

Then

$$((a+c_1)+(a+c_2))+(a+c_3)=c+(a+c_3)=a+((a+c)+(a+c_3))=$$

$$=a+\sup\{a+a+s_1+a+t_2+a+s_3:\ s_1\in S_{1k_1},\ s_3\in S_{3k_3},\ t_2\in T_{2k_2}\}=$$

$$=a+\sup\{s_1+a+t_2+a+s_3:\ s_1\in S_{1k_1},\ s_3\in S_{3k_3},\ t_2\in T_{2k_2}\},$$

$$(a+c_1)+((a+c_2)+(a+c_3))=(a+c_1)+c'=a+(c_1+c')=$$

$$=a+\sup\{s_1+a+t_2+a+s_3:\ s_1\in S_{1k_1},\ s_3\in\ S_{3k_3},t_2\in T_{2k_2}\}.$$

We have seen that $((a+c_1)+(a+c_2))+(a+c_3)=(a+c_1))+((a+c_2)+(a+c_3))$.

The remaining cases can be verified in a similar way.

Elements of $M(H)$ have inverses in $M(H)$. Let $a+c\in a+M(H)$. Then $a+(a-c+a)$ is an inverse to $a+c$ in $a+M(H)$ which completes the proof.

■

Lemma 4.7. *Let $c, c_i\in M(H)$ $(i=1,2,3)$.*

If $(c_1,c_2,c_3)\in\bar{C}_1$, then

(i$_1$) $(c_1+c,\ c_2+c,\ c_3+c)\in\bar{C}_1$,

(i$_2$) $(c+c_1,\ c+c_2,\ c+c_3)\in\bar{C}_1$,

(i$_3$) $(c_1+(a+c),\ c_2+(a+c),\ c_3+(a+c))\in\bar{C}_1$,

(i$_4$) $((a+c)+c_3,\ (a+c)+c_2,\ (a+c)+c_1))\in\bar{C}_1$.

If $(a+c_1,\ a+c_2,a+c_3)\in\bar{C}_1$, then

(ii$_1$) $((a+c_1)+c,\ (a+c_2)+c,\ (a+c_3)+c)\in\bar{C}_1$,

(ii$_2$) $(c+(a+c_1),\ c+(a+c_2),\ c+(a+c_3))\in\bar{C}_1$,

(ii$_3$) $((a+c_1)+(a+c),\ (a+c_2)+(a+c),\ (a+c_3)+(a+c))\in\bar{C}_1$,

(ii$_4$) $((a+c)+(a+c_3),\ (a+c)+(a+c_2),\ (a+c)+(a+c_1))\in\bar{C}_1$.

Proof. There are subsets $S, T, S_i,\ T_i$ of H with $c = \sup(S) = \inf(T),\ c_i = \sup(S_i) = \inf(T_i)$ $(i = 1, 2, 3)$. Then $c = \sup(S_k) = \inf(T_k), c_i = \sup(S_{ik_i}) = \inf(T_{ik_i})$ in $\bar{D}(H)$ where k, k_i are as before $(i = 1, 2, 3)$. As for $M(H)$ is an lc-group, (i_1) and (i_2) are valid.

(i_3) Let $(c_1, c_2, c_3) \in \bar{C}_1$. Consider several cases:

(α) Assume that k_1, k_2, k_3 are different elements of K_1. Then $(k_1, k_2, k_3) \in C_2$ and so $(t_1, t_2, t_3) \in C$ for each $t_i \in T_{ik_i} (i = 1, 2, 3)$. Hence $(t_1 + (a+s), t_2 + (a+s),\ t_3 + (a+s)) = (a + (a+t_1) + (a+s),\ a + (a+t_2) + (a+s), a + (a+t_3) + (a+s)) \in C$ for each $s \in S_k, t_i \in T_{ik_i} (i = 1, 2, 3)$. This yields that $(a + \sup\{a + t_1 + a + s :\ s \in S_k,\ t_1 \in T_{1k_1}\},\ a + \sup\{a + t_2 + a + s :\ s \in S_k, t_2 \in T_{2k_2}\},\ a + \sup\{a + t_3 + a + s :\ s \in S_k, t_3 \in T_{3k_3}\}) = (a + ((a + c_1) + (a + c)),\ a + ((a + c_2) + (a + c)),\ a + ((a + c_3) + (a + c))) = (c_1 + (a + c),\ c_2 + (a + c),\ c_3 + (a + c)) \in \bar{C}_1$.

(β) Let $k_1 = k_2 \neq k_3$. Then either $c_1 < c_2 < c_3$ or $c_3 < c_1 < c_2$. Assume that $c_1 < c_2 < c_3$. We have $c_1 = \inf\{t_1 \in H :\ t_1 \in T_{1k_1} \setminus T_{2k_2}\}$. Hence $t_1 < t_2 < t_3$ and so $(t_1, t_2, t_3) \in C$ for each $t_1 \in T_{1k_1} \setminus T_{2k_2}, t_2 \in T_{2k_2}, t_3 \in T_{3k_3}$. Futher we apply the same steps as in the case (α). If $c_3 < c_1 < c_2$ the proof is similar.

The cases $k_2 = k_3 \neq k_1$ and $k_3 = k_1 \neq k_2$ are analogous.

(γ) Let $k_1 = k_2 = k_3$. We have $c_1 < c_2 < c_3$ or $c_2 < c_3 < c_1$ or $c_3 < c_1 < c_2$. Suppose that $c_1 < c_2 < c_3$. From $c_1 = \inf\{t_1 \in H :\ t_1 \in T_{1k_1}, \setminus T_{2k_2}\}, c_2 = \inf\{t_2 \in H :\ t_2 \in T_{2k_2} \setminus T_{3k_3}\}$ we infer that $t_1 < t_2 < t_3$ and thus $(t_1, t_2, t_3) \in C$ for each $t_1 \in T_{1k_1} \setminus T_{2k_2}, t_2 \in T_{2k_2} \setminus T_{3k_3}, t_3 \in T_{3k_3}$. Now we apply the same procedure as in the case (α). Cases $c_2 < c_3 < c_1, c_3 < c_1 < c_2$ are analogous.

We conclude that (i_3) is satisfied.

(ii_1) Assume that $(a + c_1, a + c_2, a + c_3) \in \bar{C}_1$. Hence $(c_3, c_2, c_1) \in \bar{C}$.

According to (i_1), we get $(c_3 + c, c_2 + c, c_1 + c) \in \bar{C}$. This yields that $(a+(c_1+c), a+(c_2+c), a+(c_3+c)) = ((a+c_1)+c, (a+c_2)+c, (a+c_3)+c) \in \bar{C}_1$.

(ii_3) Again, assume that $(a+c_1, a+c_2, a+c_3) \in \bar{C}_1$. Then $(c_3, c_2, c_1) \in \bar{C}$.

With respect to (i_2), we obtain $(c_3 + (a+c), c_2 + (a+c), c_1 + (a+c)) \in \bar{C}_1$, i.e., $(a+((a+c_3)+(a+c)), a+((a+c_2)+(a+c)), a+((a+c_1)+(a+c))) \in \bar{C}_1$. Therefore, $((a+c_1)+(a+c), (a+c_2)+(a+c), (a+c_3)+(a+c)) \in \bar{C}_1$.

The remaining cases can be proved similarly. ■

From Lemmas 4.6 and 4.7 it immediately follows

Theorem 4.8. $(M_h(G); +, \bar{C}_1)$ *is a half lc-group with* $M_h(G) \uparrow = M(H)$ *and* $M_h(G) \downarrow = a + M(H)$. ■

The half *lc*-group $M_h(G)$ is said to be a *completion* of G. If $M_h(G) = G$, then G is called M_h*-complete.*

Evidently that the following lemma is valid.

Lemma 4.9. G *is* M_h*-complete if and only if* H *is* M*-complete.* ■

With respect to Theorem 3.7 and Lemma 4.9 we have:

Theorem 4.10. *Let* $H_0 \neq \{0\}$. *Then the half lc-group* $M_h(G)$ *has the following properties:*

(a_1) $M_h(G)$ *is* M_h*-complete;*

(b_1) G *is an hc-subgroup of* $M_h(G)$*;*

(c_1) *For each element* $c \in M_h(G) \uparrow$ *there exist* $k \in K_1$ *and* $S, T \subseteq H$ *such that* S_k *and* T_k *are nonempty subsets of* H *and* $c = \sup(S_k) = \inf(T_k)$ *in* $M_h(G) \uparrow$. ■

Theorem 4.11. *Let* $H_0 \neq \{0\}$. *Assume that* G' *is a half lc-group satisfying the above conditions* (a_1), (b_1) *and* (c_1) (*with* G' *instead of* $M_h(G)$). *Then there exists an isomorphism* ϕ_1 *of the half lc-group* $M_h(G)$ *onto* G' *with* $\phi_1(g) = g$ *for each* $g \in G$.

Proof. Since G' fulfils the conditions (a_1)–(c_1), $G' \uparrow$ fulfils the conditions (a)–(c) from Theorem 3.7 ($G' \uparrow$ instead of $M(H)$). Hence there exists an isomorphism ϕ of the *lc*-group $M(H)$ onto $G' \uparrow$ with $\phi(h) = h$ for each $h \in H$. For each $c \in M(H)$, we put $\phi_1(c) = \phi(c)$ and $\phi_1(a+c) = a + \phi(c)$. Therefore, ϕ_1 is an isomorphism of the half *lc*-group $M_h(G)$ onto G'. For each $h \in H$, we have $\phi_1(a+h) = a + \phi(h) = a + h$ and the proof is complete. ■

Remark. The question whether half *lc*-groups with isomorphic increasing parts are isomorphic is open.

Let a' be an element from $G\downarrow$ of the second order, $a' \neq a$. The operation $+$ and the cyclic order on the set $M'_h(G) = M(H) \cup (a' + M(H))$ are defined formally in the same way as on $M_h(G)$. It can be easily verified that the half *lc*-group $M'_h(G)$ is equal $M_h(G)$.

$M_h(G)$ and M_h-completness are defined in the same way also in the case $H_0 = \{0\}$. From Theorem 3.9 and Lemma 4.9, we infer that the following theorem holds in both cases $H_o = \{0\}$ and $H_0 \neq \{0\}$.

Theorem 4.12. *Let G be a half lc-group. Then G is M_h-complete if and only if some of the following conditions is satisfied:*

(i) *H is finite;*

(ii) *H is isomorphic to K;*

(iii) *$H_0 \neq \{0\}$ and H_0 is M-complete.*

∎

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