

**THE LATTICE OF SUBVARIETIES
OF THE BIREGULARIZATION OF THE VARIETY
OF BOOLEAN ALGEBRAS**

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Abstract

Let $\tau : F \rightarrow N$ be a type of algebras, where F is a set of fundamental operation symbols and N is the set of all positive integers. An identity $\varphi \approx \psi$ is called *biregular* if it has the same variables in each of its sides and it has the same fundamental operation symbols in each of its sides. For a variety V of type τ we denote by V_b the *biregularization* of V , i.e. the variety of type τ defined by all biregular identities from $Id(V)$.

Let B be the variety of Boolean algebras of type $\tau_b : \{+, \cdot, '\} \rightarrow N$, where $\tau_b(+)$ = $\tau_b(\cdot)$ = 2 and $\tau_b(')$ = 1. In this paper we characterize the lattice $\mathcal{L}(B_b)$ of all subvarieties of the biregularization of the variety B .

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0. PRELIMINARIES

We shall consider algebras of type $\tau : F \rightarrow N$, where F is the set of all fundamental operation symbols and N is the set of all positive integers (see [3]). If φ is a term of type τ we denote by $Var(\varphi)$ the set of all variables occurring in φ and by $F(\varphi)$ – the set of fundamental operation symbols occurring in φ . Writing $\varphi(x_{i_1}, \dots, x_{i_n})$ instead of φ we shall mean that $Var(\varphi) = \{x_{i_1}, \dots, x_{i_n}\}$. An identity $\varphi \approx \psi$ of type τ is called *regular* (see

[8] if $\text{Var}(\varphi) = \text{Var}(\psi)$. An identity $\varphi \approx \psi$ is called *biregular* if it is regular and $F(\varphi) = F(\psi)$. Regular identities and constructions connected with them were considered in [4]–[6], [8], [9], [16] and biregular identities were considered in [10]–[12], [14], [15], [18].

For a variety V of type τ we denote by $\text{Id}(V)$ the set of all identities of type τ satisfied in every algebra from V . For a variety V of type τ we denote by V_r the variety of type τ defined by all regular identities from $\text{Id}(V)$ and we denote by V_b the variety of type τ defined by the set $B(V)$ of all biregular identities from $\text{Id}(V)$. Obviously $B(V)$ is always an equational theory, so $\text{Id}(V_b) = B(V)$. The variety V_b is called the biregularization of V . We denote by $\mathcal{L}(V)$ the lattice of all subvarieties of V . Studying identities of some special structural forms is useful for examining lattices of subvarieties. Let B be the variety of Boolean algebras of type $\tau_b : \{+, \cdot, '\} \rightarrow N$, where $\tau_b(+) = \tau_b(\cdot) = 2$ and $\tau_b(') = 1$. In this paper we describe the lattice $\mathcal{L}(B_b)$.

Recall that an algebra \mathfrak{A} is subdirectly irreducible if its lattice of congruences has exactly one atom (see [7]). If an algebra \mathfrak{A} is subdirectly irreducible, we shall write shortly \mathfrak{A} is an s.i. algebra. The notation $\mathfrak{A} \simeq \mathfrak{A}'$ will stand for “ \mathfrak{A} is isomorphic to \mathfrak{A}' ”.

1. SUBDIRECTLY IRREDUCIBLE ALGEBRAS IN B_b

Let us consider the following 14 algebras of type τ_b .

$$\mathfrak{A}_1 = (\{a_1, b_1\}; +, \cdot, '), \quad \text{where } \begin{aligned} x + y &= \begin{cases} b_1, & \text{if } b_1 \in \{x, y\}, \\ a_1 & \text{otherwise,} \end{cases} \\ x \cdot y &= \begin{cases} a_1, & \text{if } a_1 \in \{x, y\}, \\ b_1 & \text{otherwise,} \end{cases} \\ a'_1 &= b_1, \quad b'_1 = a_1; \end{aligned}$$

$$\mathfrak{A}_2 = (\{a_2, b_2\}; +, \cdot, '), \quad \text{where } \begin{aligned} x + y &= \begin{cases} b_2 & \text{if } b_2 \in \{x, y\} \\ a_2 & \text{otherwise,} \end{cases} \\ x \cdot y = x' = b_2 & \text{ for every } x, y \in \{a_2, b_2\}; \end{aligned}$$

$$\mathfrak{A}_3 = (\{a_3, b_3\}; +, \cdot, '), \text{ where } x \cdot y = \begin{cases} b_3, & \text{if } b_3 \in \{x, y\}, \\ a_3 & \text{otherwise,} \end{cases}$$

$$x + y = x' = b_3 \text{ for every } x, y \in \{a_3, b_3\};$$

$$\mathfrak{A}_4 = (\{a_4, b_4\}; +, \cdot, '), \text{ where}$$

$$x + y = x \cdot y = x' = b_4 \text{ for every } x, y \in \{a_4, b_4\};$$

$$\mathfrak{A}_5 = (\{a_5, b_5\}; +, \cdot, '), \text{ where}$$

$$x + y = x \cdot y = b_5, \quad x' = x \text{ for every } x, y \in \{a_5, b_5\};$$

$$\mathfrak{A}_6 = (\{a_6, c_6, b_6\}; +, \cdot, '), \text{ where}$$

$$x + y = x \cdot y = b_6, \text{ for every } x, y \in \{a_6, c_6, b_6\},$$

$$a'_6 = c_6, \quad c'_6 = a_6, \quad b'_6 = b_6;$$

$$\mathfrak{A}_7 = (\{a_7, b_7\}; +, \cdot, '), \text{ where}$$

$$x + y = x \cdot y = \begin{cases} b_7, & \text{if } b_7 \in \{x, y\}, \\ a_7 & \text{otherwise,} \end{cases}$$

$$x' = x \text{ for every } x \in \{a_7, b_7\};$$

$$\mathfrak{A}_8 = (\{a_8, c_8, b_8\}; +, \cdot, '), \text{ where}$$

$$x + y = \begin{cases} b_8, & \text{if } b_8 \in \{x, y\}, \\ c_8, & \text{if } c_8 \in \{x, y\} \text{ and } b_8 \notin \{x, y\}, \\ a_8 & \text{otherwise,} \end{cases}$$

$$x \cdot y = \begin{cases} b_8, & \text{if } b_8 \in \{x, y\}, \\ a_8, & \text{if } a_8 \in \{x, y\} \text{ and } b_8 \notin \{x, y\}, \\ c_8 & \text{otherwise,} \end{cases}$$

$$a'_8 = c_8, \quad c'_8 = a_8, \quad b'_8 = b_8;$$

$\mathfrak{A}_9 = (\{a_9, b_9\}; +, \cdot, ')$, where

$$x + y = x \cdot y = \begin{cases} b_9, & \text{if } b_9 \in \{x, y\}, \\ a_9 & \text{otherwise,} \end{cases}$$

$$x' = b_9, \quad \text{for every } x \in \{a_9, b_9\};$$

$\mathfrak{A}_{10} = (\{a_{10}, c_{10}, b_{10}\}; +, \cdot, ')$, where

$$x + y = \begin{cases} b_{10}, & \text{if } b_{10} \in \{x, y\}, \\ c_{10}, & \text{if } c_{10} \in \{x, y\} \text{ and } b_{10} \notin \{x, y\}, \\ a_{10} & \text{otherwise,} \end{cases}$$

$$x \cdot y = \begin{cases} b_{10}, & \text{if } b_{10} \in \{x, y\}, \\ a_{10}, & \text{if } a_{10} \in \{x, y\} \text{ and } b_{10} \notin \{x, y\}, \\ c_{10} & \text{otherwise,} \end{cases}$$

$$x' = b_{10} \quad \text{for every } x \in \{a_{10}, c_{10}, b_{10}\};$$

$\mathfrak{A}_{11} = (\{a_{11}, b_{11}\}; +, \cdot, ')$, where

$$x + y = \begin{cases} b_{11}, & \text{if } b_{11} \in \{x, y\}, \\ a_{11} & \text{otherwise,} \end{cases}$$

$$x \cdot y = b_{11} \quad \text{for every } x, y \in \{a_{11}, b_{11}\},$$

$$x' = x \quad \text{for every } x \in \{a_{11}, b_{11}\};$$

$\mathfrak{A}_{12} = (\{a_{12}, c_{12}, b_{12}\}; +, \cdot, ')$, where

$$x + y = \begin{cases} b_{12}, & \text{if } b_{12} \in \{x, y\}, \\ c_{12}, & \text{if } c_{12} \in \{x, y\} \text{ and } b_{12} \notin \{x, y\}, \\ a_{12} & \text{otherwise,} \end{cases}$$

$$x \cdot y = b_{12} \text{ for every } x, y \in \{a_{12}, c_{12}, b_{12}\},$$

$$a'_{12} = c_{12}, \quad c'_{12} = a_{12}, \quad b'_{12} = b_{12};$$

$\mathfrak{A}_{13} = (\{a_{13}, b_{13}\}; +, \cdot, ')$, where

$$x + y = b_{13} \text{ for every } x, y \in \{a_{13}, b_{13}\},$$

$$x \cdot y = \begin{cases} b_{13}, & \text{if } b_{13} \in \{x, y\}, \\ a_{13} & \text{otherwise,} \end{cases}$$

$$x' = x \text{ for every } x \in \{a_{13}, b_{13}\};$$

$\mathfrak{A}_{14} = (\{a_{14}, c_{14}, b_{14}\}; +, \cdot, ')$, where

$$x + y = b_{14} \text{ for every } x, y \in \{a_{14}, c_{14}, b_{14}\},$$

$$x \cdot y = \begin{cases} b_{14}, & \text{if } b_{14} \in \{x, y\}, \\ a_{14}, & \text{if } a_{14} \in \{x, y\} \text{ and } b_{14} \notin \{x, y\}, \\ c_{14} & \text{otherwise,} \end{cases}$$

$$a'_{14} = c_{14}, \quad c'_{14} = a_{14}, \quad b'_{14} = b_{14}.$$

It is easy to check that none two of above 14 algebras are isomorphic.

Theorem 1.1. *Let $\mathfrak{A} = (A; +, \cdot, ')$ be an algebra of type τ_b . Then \mathfrak{A} is subdirectly irreducible and belongs to B_b if and only if \mathfrak{A} is isomorphic to one of the algebras $\mathfrak{A}_1, \dots, \mathfrak{A}_{14}$.*

Proof. For varieties K_1, \dots, K_n of the same type we denote by $K_1 \otimes \dots \otimes K_n$ the class of all algebras isomorphic to a subdirect product of a family $\{\mathfrak{A}_1, \dots, \mathfrak{A}_n\}$ of algebras, where \mathfrak{A}_i runs over K_i for every $i = 1, \dots, n$.

For $\tilde{F} \subseteq \{+, \cdot, '\}$, we denote by $B_{\tilde{F}}$ the variety of type τ_b satisfying all regular identities $\varphi \approx \psi$ from $Id(B)$ with $F(\varphi) \cup F(\psi) \subseteq \tilde{F}$ and satisfying all identities of type τ_b such that $F(\varphi) \cap (\{+, \cdot, '\} \setminus \tilde{F}) \neq \emptyset \neq F(\psi) \cap (\{+, \cdot, '\} \setminus \tilde{F})$. It was proved in [12], Theorem 9, that

$$(1.1) \quad B_b = B_r \otimes B_{\{+, \cdot\}} \otimes B_{\{+, '\}} \otimes B_{\{\cdot, '\}} \otimes B_{\{+\}} \otimes B_{\{\cdot\}} \otimes B_{\{'\}} \otimes B_\emptyset$$

Consequently to find all subdirectly irreducible algebras from B_b it is enough to find all s.i. algebras from the varieties of the right side of (1.1).

It was proved in [6] that \mathfrak{A} is s.i. and $\mathfrak{A} \in B_r$ iff \mathfrak{A} is isomorphic to one of the algebras $\mathfrak{A}_1, \mathfrak{A}_7$ or \mathfrak{A}_8 . It was proved in [13] that \mathfrak{A} is s.i. and belongs to $B_{\{+\}}$ iff \mathfrak{A} is isomorphic to \mathfrak{A}_2 ; \mathfrak{A} is s.i. and $\mathfrak{A} \in B_{\{\cdot\}}$ iff $\mathfrak{A} \simeq \mathfrak{A}_3$; \mathfrak{A} is s.i. and $\mathfrak{A} \in B_\emptyset$ iff $\mathfrak{A} \simeq \mathfrak{A}_4$ (cf. also [2]); \mathfrak{A} is s.i. and $\mathfrak{A} \in B_{\{'\}}$ iff $\mathfrak{A} \simeq \mathfrak{A}_5$ or $\mathfrak{A} \simeq \mathfrak{A}_6$. It was proved in [19] (see Section 3, Examples 3.3–3.5) that \mathfrak{A} is s.i. and $\mathfrak{A} \in B_{\{+, \cdot\}}$ iff $\mathfrak{A} \simeq \mathfrak{A}_9$ or $\mathfrak{A} \simeq \mathfrak{A}_{10}$ and $\mathfrak{A} \in B_{\{+, '\}}$ iff $\mathfrak{A} \simeq \mathfrak{A}_{11}$ or $\mathfrak{A} \simeq \mathfrak{A}_{12}$; \mathfrak{A} is s.i. and $\mathfrak{A} \in B_{\{\cdot, '\}}$ iff $\mathfrak{A} \simeq \mathfrak{A}_{13}$ or $\mathfrak{A} \simeq \mathfrak{A}_{14}$. ■

2. THE LATTICE OF SUBVARIETIES OF B_b

Denote $Ir(B_b) = \{\mathfrak{A}_1, \dots, \mathfrak{A}_{14}\}$. For a variety $V \subseteq B_b$ we denote $Ir(V) = \{\mathfrak{A}_k \in Ir(B_b) : \mathfrak{A}_k \in V\}$. Consequently, to describe the lattice $\mathcal{L}(B_b)$ we have to find all subsets T of $Ir(B_b)$ being of the form $Ir(V)$ for some $V \subseteq B_b$. Apriori we have 2^{14} possibilities. However due to the lemmas below we can essentially reduce this amount.

Lemma 2.1. $\mathfrak{A}_1 \in HSP(\mathfrak{A}_8)$.

Proof. Observe that the subalgebra $(\{a_8, c_8\}; \{+, \cdot, '\}|_{\{a_8, c_8\}})$ of \mathfrak{A}_8 is isomorphic to \mathfrak{A}_1 . ■

Lemma 2.2. $\mathfrak{A}_{2n-1} \in HSP(\mathfrak{A}_{2n})$ for $3 \leq n \leq 7$.

Proof. Put $h(a_{2n}) = h(c_{2n}) = a_{2n-1}$, $h(b_{2n}) = b_{2n-1}$. Thus h is a homomorphism. ■

Lemma 2.3. $\mathfrak{A}_{2n} \in HSP(\{\mathfrak{A}_1, \mathfrak{A}_{2n-1}\})$ for $3 \leq n \leq 7$.

Proof. In the direct product $\mathfrak{A}_1 \times \mathfrak{A}_{2n-1}$ put $h(\langle a_1, a_{2n-1} \rangle) = a_{2n}$, $h(\langle b_1, a_{2n-1} \rangle) = c_{2n}$, $h(\langle a_1, b_{2n-1} \rangle) = h(\langle b_1, b_{2n-1} \rangle) = b_{2n}$. ■

Lemma 2.4. $\mathfrak{A}_2 \in HSP(\{\mathfrak{A}_9, \mathfrak{A}_{11}\})$.

Proof. In the direct product $\mathfrak{A}_9 \times \mathfrak{A}_{11}$ put $h(\langle a_9, a_{11} \rangle) = a_2$, $h(\langle a_9, b_{11} \rangle) = h(\langle b_9, a_{11} \rangle) = h(\langle b_9, b_{11} \rangle) = b_2$. ■

Lemma 2.5. $\mathfrak{A}_3 \in HSP(\{\mathfrak{A}_9, \mathfrak{A}_{13}\})$.

Proof. The proof is analogous to that of Lemma 2.4. ■

Lemma 2.6. $\mathfrak{A}_5 \in HSP(\{\mathfrak{A}_{11}, \mathfrak{A}_{13}\})$.

Proof. The proof is analogous to that of Lemma 2.4. ■

Lemma 2.7. $\mathfrak{A}_6 \in HSP(\{\mathfrak{A}_5, \mathfrak{A}_{12}\})$.

Proof. In the direct product $\mathfrak{A}_5 \times \mathfrak{A}_{12}$ put $h(\langle a_5, a_{12} \rangle) = a_6$, $h(\langle a_5, c_{12} \rangle) = c_6$, $h(\langle x, y \rangle) = b_6$ otherwise. ■

Lemma 2.8. $\mathfrak{A}_6 \in HSP(\{\mathfrak{A}_5, \mathfrak{A}_{14}\})$.

Proof. The proof is analogous to that of Lemma 2.7. ■

Lemma 2.9. $\mathfrak{A}_6 \in HSP(\{\mathfrak{A}_{11}, \mathfrak{A}_{14}\})$.

Proof. In the direct product $\mathfrak{A}_{11} \times \mathfrak{A}_{14}$ put $h(\langle a_{11}, a_{14} \rangle) = a_6$, $h(\langle a_{11}, c_{14} \rangle) = c_6$ and $h(\langle x, y \rangle) = b_6$ otherwise. ■

Lemma 2.10. $\mathfrak{A}_6 \in HSP(\{\mathfrak{A}_{12}, \mathfrak{A}_{13}\})$.

Proof. The proof is analogous to that of Lemma 2.9. ■

Lemma 2.11. \mathfrak{A}_4 belongs to each of the sets $HSP(\{\mathfrak{A}_2, \mathfrak{A}_3\})$, $HSP(\{\mathfrak{A}_2, \mathfrak{A}_5\})$, $HSP(\{\mathfrak{A}_3, \mathfrak{A}_5\})$, $HSP(\{\mathfrak{A}_2, \mathfrak{A}_{13}\})$, $HSP(\{\mathfrak{A}_3, \mathfrak{A}_{11}\})$, $HSP(\{\mathfrak{A}_5, \mathfrak{A}_9\})$.

Proof. The proof is easy and it is left to the reader. ■

A set $T \subseteq Ir(B_b)$ will be called B_b -closed or briefly closed if it satisfies the following conditions (c₁)–(c₁₁):

- (c₁) if $\mathfrak{A}_8 \in T$, then $\mathfrak{A}_1 \in T$;
- (c₂) if $3 \leq n \leq 7$ and $\mathfrak{A}_{2n} \in T$, then $\mathfrak{A}_{2n-1} \in T$;
- (c₃) if $3 \leq n \leq 7$ and $\{\mathfrak{A}_1, \mathfrak{A}_{2n-1}\} \subseteq T$, then $\mathfrak{A}_{2n} \in T$;
- (c₄) if $\{\mathfrak{A}_9, \mathfrak{A}_{11}\} \subseteq T$, then $\mathfrak{A}_2 \in T$;
- (c₅) if $\{\mathfrak{A}_9, \mathfrak{A}_{13}\} \subseteq T$, then $\mathfrak{A}_3 \in T$;
- (c₆) if $\{\mathfrak{A}_{11}, \mathfrak{A}_{13}\} \subseteq T$, then $\mathfrak{A}_5 \in T$;
- (c₇) if $\{\mathfrak{A}_5, \mathfrak{A}_{12}\} \subseteq T$, then $\mathfrak{A}_6 \in T$;
- (c₈) if $\{\mathfrak{A}_5, \mathfrak{A}_{14}\} \subseteq T$, then $\mathfrak{A}_6 \in T$;
- (c₉) if $\{\mathfrak{A}_{11}, \mathfrak{A}_{14}\} \subseteq T$, then $\mathfrak{A}_6 \in T$;
- (c₁₀) if $\{\mathfrak{A}_{12}, \mathfrak{A}_{13}\} \subseteq T$, then $\mathfrak{A}_6 \in T$;
- (c₁₁) $\left\{ \begin{array}{l} \text{if } \{\mathfrak{A}_2, \mathfrak{A}_3\} \subseteq T, \text{ then } \mathfrak{A}_4 \in T; \quad \text{if } \{\mathfrak{A}_2, \mathfrak{A}_5\} \subseteq T, \text{ then } \mathfrak{A}_4 \in T; \\ \text{if } \{\mathfrak{A}_3, \mathfrak{A}_5\} \subseteq T, \text{ then } \mathfrak{A}_4 \in T; \quad \text{if } \{\mathfrak{A}_2, \mathfrak{A}_{13}\} \subseteq T, \text{ then } \mathfrak{A}_4 \in T; \\ \text{if } \{\mathfrak{A}_3, \mathfrak{A}_{11}\} \subseteq T, \text{ then } \mathfrak{A}_4 \in T; \quad \text{if } \{\mathfrak{A}_5, \mathfrak{A}_9\} \subseteq T, \text{ then } \mathfrak{A}_4 \in T. \end{array} \right.$

Lemma 2.12. *If $T \subseteq Ir(B_b)$, T is B_b -closed and $\mathfrak{A}_k \notin T$ for some $k \in \{1, \dots, 14\}$, then $\mathfrak{A}_k \notin HSP(T)$.*

Proof. Let $k = 1$. Then $T \subseteq \{\mathfrak{A}_2, \dots, \mathfrak{A}_{14}\}$. By (c₁) $\mathfrak{A}_8 \notin T$. Thus $T \subseteq \{\mathfrak{A}_2, \dots, \mathfrak{A}_{14}\} \setminus \{\mathfrak{A}_8\}$. Take the identity

$$(2.1) \quad (((x+y) \cdot (x+y))')' \approx (((x \cdot y) + (x \cdot y))')'$$

Then we check that (2.1) is satisfied in every algebra \mathfrak{A}_i for $i \in \{2, \dots, 14\} \setminus \{8\}$, so (2.1) is satisfied in $HSP(T)$ but (2.1) is not satisfied in \mathfrak{A}_1 . Consequently $\mathfrak{A}_1 \notin HSP(T)$.

Let $k = 2$. Then none of the sets $\{\mathfrak{A}_9, \mathfrak{A}_{11}\}$, $\{\mathfrak{A}_9, \mathfrak{A}_{12}\}$, $\{\mathfrak{A}_{10}, \mathfrak{A}_{11}\}$, $\{\mathfrak{A}_{10}, \mathfrak{A}_{12}\}$ is included in T . In fact, by (c₂), if one of the sets is included in T , then $\{\mathfrak{A}_9, \mathfrak{A}_{11}\} \subseteq T$ and by (c₄) $\mathfrak{A}_2 \in T$, a contradiction. So, it must be

$$(2.2) \quad T \cap \{\mathfrak{A}_{11}, \mathfrak{A}_{12}\} = \emptyset$$

or

$$(2.3) \quad T \cap \{\mathfrak{A}_9, \mathfrak{A}_{10}\} = \emptyset.$$

If (2.2) holds, then take the identity

$$x + x \approx (x + x) \cdot (x + x).$$

Then every algebra from T satisfies this identity, so it is satisfied in $HSP(T)$ but \mathfrak{A}_2 does not satisfy it. In case (2.3) we take the identity

$$x + x \approx ((x + x)')'.$$

Let $k = 3$. Then by (c₅) and (c₂) none of the sets $\{\mathfrak{A}_9, \mathfrak{A}_{13}\}$, $\{\mathfrak{A}_9, \mathfrak{A}_{14}\}$, $\{\mathfrak{A}_{10}, \mathfrak{A}_{13}\}$, $\{\mathfrak{A}_{10}, \mathfrak{A}_{14}\}$ can be included in T . If $T \cap \{\mathfrak{A}_{13}, \mathfrak{A}_{14}\} = \emptyset$, we take the identity

$$x \cdot x \approx (x \cdot x) + (x \cdot x).$$

If $T \cap \{\mathfrak{A}_9, \mathfrak{A}_{10}\} = \emptyset$, we take the identity

$$((x \cdot x)')' \approx x \cdot x.$$

Let $k = 4$. By (c₂) and (c₁₁) T must be included in one of the sets:

$$\{\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_7, \mathfrak{A}_8, \mathfrak{A}_9, \mathfrak{A}_{10}, \mathfrak{A}_{11}, \mathfrak{A}_{12}\}, \{\mathfrak{A}_1, \mathfrak{A}_3, \mathfrak{A}_7, \mathfrak{A}_8, \mathfrak{A}_9, \mathfrak{A}_{10}, \mathfrak{A}_{13}, \mathfrak{A}_{14}\}, \\ \{\mathfrak{A}_1, \mathfrak{A}_5, \mathfrak{A}_6, \mathfrak{A}_7, \mathfrak{A}_8, \mathfrak{A}_{11}, \mathfrak{A}_{12}, \mathfrak{A}_{13}, \mathfrak{A}_{14}\}.$$

We take the identities $x + x \approx x$, $x \cdot x \approx x$, $(x')' \approx x$, respectively.

Let $k = 5$. By (c₂), $\mathfrak{A}_6 \notin T$ and, by (c₆) and (c₂), none of the sets $\{\mathfrak{A}_{11}, \mathfrak{A}_{13}\}$, $\{\mathfrak{A}_{11}, \mathfrak{A}_{14}\}$, $\{\mathfrak{A}_{12}, \mathfrak{A}_{13}\}$, $\{\mathfrak{A}_{12}, \mathfrak{A}_{14}\}$ is included in T . If $T \cap \{\mathfrak{A}_{13}, \mathfrak{A}_{14}\} = \emptyset$, we take

$$(x')' \approx (x')' + (x')'.$$

If $T \cap \{\mathfrak{A}_{11}, \mathfrak{A}_{12}\} = \emptyset$, we take the identity

$$(x')' \approx (x')' \cdot (x')'.$$

Let $k = 6$. If $\mathfrak{A}_5 \in T$, then $T \cap \{\mathfrak{A}_8, \mathfrak{A}_1, \mathfrak{A}_{12}, \mathfrak{A}_{14}\} = \emptyset$ by (c₃), (c₁), (c₇), (c₈). We take the identity $(x')' \approx x'$. Let $\mathfrak{A}_5 \notin T$. If $\mathfrak{A}_1 \in T$, then, by (c₂), (c₃), (c₉), (c₁₀), it must be

$$(2.4) \quad T \cap \{\mathfrak{A}_{13}, \mathfrak{A}_{14}\} = \emptyset$$

or

$$(2.5) \quad T \cap \{\mathfrak{A}_{11}, \mathfrak{A}_{12}\} = \emptyset.$$

If (2.4) holds, we take the identity

$$(x')' \approx ((x+x)')$$

If (2.5) holds, we take the identity

$$(x')' \approx ((x \cdot x)')$$

If $\mathfrak{A}_5 \notin T$ and $\mathfrak{A}_1 \notin T$, then $\mathfrak{A}_8 \notin T$ by (c₁). Then, by (c₉), (c₁₀) and (c₆) we have two possibilities: (2.4), (2.5). We take the identities $(x')' \approx ((x+x)')$, $(x')' \approx ((x \cdot x)')$, respectively.

Let $k = 7$. Then by (c₂) $\mathfrak{A}_8 \notin T$. We take the identity

$$(2.6) \quad (((x+(x \cdot y)))')' \approx (((x+(x \cdot z)))')$$

Let $k = 8$. Then T does not contain both \mathfrak{A}_1 and \mathfrak{A}_7 by (c₃). If $\mathfrak{A}_7 \notin T$, we take the identity (2.6). If $\mathfrak{A}_1 \notin T$ we take the identity (2.1).

Let $k = 9$. Then $\mathfrak{A}_{10} \notin T$ by (c₂). We take

$$(2.7) \quad (((x+y) \cdot (x+y))')' \approx (x+y) \cdot (x+y).$$

Let $k = 10$. If $\mathfrak{A}_9 \notin T$, then we take the identity (2.7). If $\mathfrak{A}_9 \in T$, then $\{\mathfrak{A}_1, \mathfrak{A}_8\} \not\subseteq T$ by (c₃) and (c₁). We take

$$(x \cdot y) + (x \cdot y) \approx (x+y) \cdot (x+y).$$

Let $k = 11$. Then $\mathfrak{A}_{12} \notin T$ by (c₂). We take

$$(2.8) \quad (((x+y) \cdot (x+y))')' \approx ((x+y)')$$

Let $k = 12$. If $\mathfrak{A}_{11} \notin T$, then we take the identity (2.8). If $\mathfrak{A}_{11} \in T$, then $\mathfrak{A}_1, \mathfrak{A}_8 \notin T$ by (c₃) and (c₁). Then we take

$$((x+y)')' \approx (x+y)'$$

Let $k = 13$. Then $\mathfrak{A}_{14} \notin T$ by (c_2) . We take

$$(2.9) \quad (((x \cdot y) + (x \cdot y))')' \approx ((x \cdot y)')$$

Let $k = 14$. If $\mathfrak{A}_{13} \notin T$, we take the identity (2.9). If $\mathfrak{A}_{13} \in T$, then $\mathfrak{A}_1, \mathfrak{A}_8 \notin T$ by (c_3) and (c_1) . We take

$$((x \cdot y)')' \approx (x \cdot y)'$$

■

Lemma 2.13. *If a variety V belongs to $\mathcal{L}(B_b)$ and $\mathfrak{A} \in V$, then \mathfrak{A} is isomorphic to a subdirect product of a family of subdirectly irreducible algebras belonging to $Ir(V)$.*

Proof. By Birkhoff's Subdirect Representation Theorem (see [1]), if $\mathfrak{A} \in V$, then it is isomorphic to an algebra \mathfrak{A}' being a subdirect product of a family $\{\mathfrak{A}_j\}_{j \in J}$ of subdirectly irreducible algebras from V . By Theorem 1.1, each \mathfrak{A}_j is isomorphic to an algebra \mathfrak{A}_j^* from $Ir(B_b)$. Thus \mathfrak{A}_j^* belongs to V and belongs to $Ir(B_b)$, hence \mathfrak{A}_j^* belongs to $Ir(V)$. Consequently, \mathfrak{A}' is isomorphic to an algebra \mathfrak{A}^* being a subdirect product of the family $\{\mathfrak{A}_j^*\}_{j \in J}$ and \mathfrak{A} is isomorphic to \mathfrak{A}^* . ■

We denote by $\mathbf{T}(B_b)$ the set of all B_b -closed sets.

Lemma 2.14. *We have:*

- (i) *For every variety $V \in \mathcal{L}(B_b)$, the set $Ir(V)$ is B_b -closed;*
- (ii) *For every variety $V \in \mathcal{L}(B_b)$, we have $V = HSP(Ir(V))$;*
- (iii) *If $T \in \mathbf{T}(B_b)$, then $T = Ir(HSP(T))$;*
- (iv) *If $V_1, V_2 \in \mathcal{L}(B_b)$, then $V_1 \subseteq V_2$ iff $Ir(V_1) \subseteq Ir(V_2)$.*

Proof. (i): If $\mathfrak{A}_8 \in Ir(V)$, then, by Lemma 2.1, we have $\mathfrak{A}_1 \in HSP(\mathfrak{A}_8) \subseteq HSP(Ir(V)) \subseteq V$, but $\mathfrak{A}_1 \in Ir(B_b)$, so $\mathfrak{A}_1 \in V \cap Ir(B_b) = Ir(V)$. Consequently, the set $Ir(V)$ satisfies (c_1) . Similarly, using Lemmas 2.2–2.11, we show that $Ir(V)$ satisfies (c_2) – (c_{11}) .

(ii): Since $Ir(V) \subseteq V$, $HSP(Ir(V)) \subseteq V$. The converse inclusion follows at once from Lemma 2.13.

(iii): If an algebra \mathfrak{A} belongs to T , then $\mathfrak{A} \in HSP(T)$. But $\mathfrak{A} \in Ir(B_b)$ since $T \subseteq Ir(B_b)$, so $\mathfrak{A} \in Ir(HSP(T))$. If $\mathfrak{A} \notin T$, then $\mathfrak{A} \notin HSP(T)$ by Lemma 2.12, hence $\mathfrak{A} \notin Ir(HSP(T))$.

(iv): If $V_1 \subseteq V_2$, then $Ir(V_1) \subseteq Ir(V_2)$ by the definition of $Ir(V)$. The converse implication follows at once from Lemma 2.13. ■

Theorem 2.15. *The set $T \subseteq Ir(B_b)$ is equal to $Ir(V)$ for some variety $V \in \mathcal{L}(B_b)$ iff T is B_b -closed. There are 490 B_b -closed sets.*

Proof. The first statement follows from Lemma 2.14 (i) and (iii).

Using a computer and transforming our considerations to indices of algebras \mathfrak{A}_k from $Ir(B_b)$ one can find out $|\mathbf{T}(B_b)| = 490$. ■

Theorem 2.16. *The lattice $(\mathcal{L}(B_b); \subseteq)$ as a poset is isomorphic to the poset $(\mathbf{T}(B_b); \subseteq)$. Therefore the lattice $(\mathcal{L}(B_b); \subseteq)$ is isomorphic to the lattice $(\mathbf{T}(B_b); \subseteq)$ and $\text{card}(\mathcal{L}(B_b)) = 490$.*

Proof. For $V \in \mathcal{L}(B_b)$ put $\varphi(V) = Ir(V)$. Then φ is well defined by the definition of $Ir(V)$ and, by Lemma 2.14 (i), φ maps $\mathcal{L}(B_b)$ into $\mathbf{T}(B_b)$. If $Ir(V_1) = Ir(V_2)$, then, by Lemma 2.14 (ii), $V_1 = HSP(Ir(V_1)) = HSP(Ir(V_2)) = V_2$. Thus φ is 1-1. By Lemma 2.14 (iii), φ is onto. If $V_1 \subseteq V_2$, then $Ir(V_1) \subseteq Ir(V_2)$ by the definition of $Ir(V)$. The converse inclusion follows at once from Lemma 2.13. ■

Remark 2.17. Results of this paper were presented on the conference “9th Workshop in Mathematics”, organized by Technical University of Zielona Góra at September 2001, in Gronów (Poland).

The statement $\text{card}(\mathcal{L}(B_b)) = 490$ was confirmed on this conference by Peter Burmeister (Darmstadt, Germany) using his *ConImp* computer program based on the Formal Concept Analysis. For the documentation of the program see P. Burmeister, *ConImp – Ein Programm zur Formalen Begriffsanalyse* in: G. Stumme and R. Wille (Eds.), *Begriffliche Wissensverarbeitung: Methoden und Anwendungen*, Springer-Verlag, Berlin 2000, pp. 25–56; extended English version – *Formal Concept Analysis with ConImp: Introduction to the basic features* – one can find on the WWW-server: http://www.mathematik.tu-darmstadt.de/ags/ag1/Software/software_de.html

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