

CONGRUENCE CLASSES IN BROUWERIAN SEMILATTICES¹

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Abstract

Brouwerian semilattices are meet-semilattices with 1 in which every element a has a relative pseudocomplement with respect to every element b , i. e. a greatest element c with $a \wedge c \leq b$. Properties of classes of reflexive and compatible binary relations, especially of congruences of such algebras are described and an abstract characterization of congruence classes via ideals is obtained.

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1. INTRODUCTION

Definition 1.1. Let (S, \wedge) be a meet-semilattice and $a, b, c \in S$ and let \leq denote its induced partial ordering relation. The element c is called a *relative pseudocomplement* of a with respect to b if c is the greatest element x of S satisfying $a \wedge x \leq b$. An algebra $(S, \wedge, *, 1)$ of type $(2, 2, 0)$ is called a *Brouwerian semilattice* if (S, \wedge) is a meet-semilattice with greatest element 1 and, for every $a, b \in S$, $a * b$ is the relative pseudocomplement of a with respect to b .

Remark 1.1. Without loss of generality the greatest element 1 of (S, \leq) can be included in the similarity type of a Brouwerian semilattice since it is an algebraic (i. e. an equationally definable) constant, namely $a * a = 1$ for each $a \in S$ (see Lemma 1.1).

Notational convention. Throughout the paper let $\mathcal{S} = (S, \wedge, *, 1)$ denote an arbitrary but fixed Brouwerian semilattice.

Remark 1.2. It is well-known that the class of all Brouwerian semilattices forms a variety.

Lemma 1.1. For $a, b, c \in S$ (i)–(xii) hold:

- (i) $a * 1 = 1$;
- (ii) $1 * a = a$;
- (iii) $a * a = 1$;
- (iv) $(a * a) * a = a$;
- (v) $a \leq b * a$;
- (vi) $a \wedge (a * b) = a \wedge b$;
- (vii) $a \leq b$ if and only if $a * b = 1$;
- (viii) if $b \leq c$, then $c * a \leq b * a$;
- (ix) $b \leq (b * a) * a$;
- (x) $((b * a) * a) * a = b * a$;
- (xi) $((b \wedge c) * a) * a = ((b * a) * a) \wedge ((c * a) * a)$;
- (xii) $(b \wedge c) * a = (b \wedge ((c * a) * a)) * a = (((b * a) * a) \wedge c) * a =$
 $= (((b * a) * a) \wedge ((c * a) * a)) * a$.

Remark 1.3. Though the listed properties of Brouwerian semilattices are mostly known (cf. e. g. [3]), for the convenience of the reader we provide a proof.

Proof of Lemma 1.1. (i)–(iii) are trivial.

(iv) follows from (iii) and (ii).

(v) follows from $b \wedge a \leq a$.

(vi): Since $a \wedge (a * b) \leq b$, it holds $a \wedge (a * b) \leq a \wedge b$. On the other hand (v) implies $a \wedge b \leq a \wedge (a * b)$.

(vii): If $a \leq b$, then $a * b = 1$. If, conversely, $a * b = 1$, then $a = a \wedge 1 = a \wedge (a * b) = a \wedge b \leq b$ according to (vi).

(viii): $b \wedge (c * a) \leq c \wedge (c * a) \leq a$ and, hence, $c * a \leq b * a$.

(ix): $(b * a) \wedge b = b \wedge (b * a) \leq a$ and, hence, $b \leq (b * a) * a$.

(x): $((b * a) * a) * a \leq b * a$ according to (ix) and (viii). On the other hand $b * a \leq ((b * a) * a) * a$ according to (ix).

(xi): From $b \wedge c \leq b, c$, it follows by applying (viii) twice $((b \wedge c) * a) * a \leq (b * a) * a, (c * a) * a$ and, hence, $((b \wedge c) * a) * a \leq ((b * a) * a) \wedge ((c * a) * a)$. On the other hand the following are equivalent:

$$b \wedge c \wedge ((b \wedge c) * a) \leq a,$$

$$c \wedge ((b \wedge c) * a) \leq b * a,$$

$$c \wedge ((b \wedge c) * a) \leq ((b * a) * a) * a,$$

$$c \wedge ((b \wedge c) * a) \wedge ((b * a) * a) \leq a,$$

$$((b \wedge c) * a) \wedge ((b * a) * a) \leq c * a,$$

$$((b \wedge c) * a) \wedge ((b * a) * a) \leq ((c * a) * a) * a,$$

$$((b \wedge c) * a) \wedge ((b * a) * a) \wedge ((c * a) * a) \leq a \text{ and}$$

$$((b * a) * a) \wedge ((c * a) * a) \leq ((b \wedge c) * a) * a.$$

(xii): According to (x) and (xi) one obtains

$$(b \wedge c) * a = (((b \wedge c) * a) * a) * a = (((b * a) * a) \wedge ((c * a) * a)) * a,$$

$$\begin{aligned} (b \wedge ((c * a) * a)) * a &= (((b \wedge ((c * a) * a)) * a) * a) * a = \\ &= (((b * a) * a) \wedge (((c * a) * a) * a)) * a = \\ &= (((b * a) * a) \wedge ((c * a) * a)) * a \end{aligned}$$

and

$$\begin{aligned} (((b * a) * a) \wedge c) * a &= (((((b * a) * a) \wedge c) * a) * a) * a = \\ &= (((((b * a) * a) * a) * a) \wedge ((c * a) * a)) * a = \\ &= (((b * a) * a) \wedge ((c * a) * a)) * a. \end{aligned}$$

■

Remark 1.4. In the following we often make use of Lemma 1.1 without explicitly mentioning it.

2. REFLEXIVE AND COMPATIBLE BINARY RELATIONS IN BROUWERIAN SEMILATTICES

Let R be a binary relation on S , $a \in R$ and n a positive integer. Then $[a]R := \{x \in S \mid x R a\}$, $R^n := R \circ R \circ \dots \circ R$ with n factors and R is called *compatible with respect to \mathcal{S}* if it has the substitution property with respect to both operations \wedge as well as $*$.

In this section R, R_1, R_2 denote some arbitrary but fixed reflexive binary relations on S which are compatible with respect to \mathcal{S} .

Theorem 2.1. *If $a \in S$ is the least element of $[a]R_1$, then $[a](R_2 \circ R_1)^n \subseteq [a](R_1 \circ R_2)^n$ and $[a](R_1 \circ (R_2 \circ R_1)^n) \subseteq [a](R_2 \circ (R_1 \circ R_2)^n)$ for every positive integer n .*

Proof. Let n be a positive integer. If $b \in [a](R_2 \circ R_1)^n$, then there exist $a_1, \dots, a_{2n} \in S$ with $b R_2 a_1 R_1 \dots R_1 a_{2n} = a$ and hence

$$b = b \wedge 1 = b \wedge (a_1 * a_1) R_1 b \wedge (a_1 * a) R_2 a_1 \wedge (a_1 * a) = a_1 \wedge a = a$$

for $n = 1$ and

$$\begin{aligned}
 b &= b \wedge 1 = b \wedge (a_1 * a_1) R_1 b \wedge (a_1 * a_2) R_2 a_1 \wedge (a_1 * a_3) = a_1 \wedge a_3 R_1 \dots \\
 &\dots R_1 a_{2n-2} \wedge a_{2n} R_2 a_{2n-1} \wedge a = a
 \end{aligned}$$

for $n > 1$, and therefore $b \in [a](R_1 \circ R_2)^n$. If $b \in [a](R_1 \circ (R_2 \circ R_1)^n)$, then there exist $a_1, \dots, a_{2n+1} \in S$ with $b R_1 a_1 R_2 \dots R_2 a_{2n} R_1 a_{2n+1} = a$ and hence

$$\begin{aligned}
 b &= b \wedge 1 = b \wedge (a_1 * a_1) R_2 b \wedge (a_1 * a_2) R_1 a_1 \wedge (a_1 * a_3) = a_1 \wedge a_3 R_2 \dots \\
 &\dots R_2 a_{2n-2} \wedge a_{2n} R_1 a_{2n-1} \wedge a R_2 a_{2n} \wedge a = a,
 \end{aligned}$$

and therefore $b \in [a](R_2 \circ (R_1 \circ R_2)^n)$. ■

Lemma 2.1. *If $a \in S$ is the least element of $[a]R$, then $[a]R \subseteq [a]R^{-1}$.*

Proof. If $b \in [a]R$, then one obtains

$$b = b \wedge 1 = b \wedge (a * a) \in [b \wedge (b * a)]R^{-1} = [b \wedge a]R^{-1} = [a]R^{-1}.$$
■

Remark 2.1. From Lemma 2.1 it follows that if $a \in S$ is the least element of both $[a]R$ and $[a]R^{-1}$, then $[a]R = [a]R^{-1}$.

Lemma 2.2. *If $a \in S$ is the least element of $[a]R$, then $[a]R^n = [a]R$ for every positive integer n .*

Proof. We use induction on n . The case $n = 1$ is trivial. Now assume $n \geq 1$ and $[a]R^n = [a]R$. The inclusion $[a]R \subseteq [a]R^{n+1}$ is trivial. If $b \in [a]R^{n+1}$, then there exist $c_1, \dots, c_{n+1} \in S$ with $b R c_1 R \dots R c_n R c_{n+1} = a$ and therefore

$$b = b \wedge 1 = b \wedge (c_1 * c_1) R c_1 \wedge (c_1 * c_2) = c_1 \wedge c_2 R \dots R c_n \wedge a = a,$$

whence $b \in [a]R^n$ which implies $[a]R^{n+1} \subseteq [a]R^n = [a]R$. ■

Definition 2.1. For every subset M of S^2 let $\Theta(M)$ denote the least congruence on S including M . $\Theta(M)$ is usually called the *congruence generated by M* .

Theorem 2.2. *If $a \in S$ is the least element of both $[a]R$ and $[a]R^{-1}$, then $[a]R = [a]\Theta(R)$.*

Proof. Let $b \in [a](R \circ R^{-1})$. Then there exists an element c of S with $bRcR^{-1}a$. Since a is the least element of $[a]R^{-1}$ it follows $c \geq a$. This implies $b \wedge aRc \wedge a = a$ and since a is the least element of $[a]R$, it follows $b \wedge a \geq a$. This shows $b \geq a$. Therefore, a is the least element of $[a](R \circ R^{-1})$ and it follows from Lemma 2.2 that $[a](R \circ R^{-1})^n = [a](R \circ R^{-1})$ for every positive integer n . Because of Remark 2.1 and Lemma 2.2, $[a]R = [a]R^{-1}$ and $[a](R \circ R) = [a]R$, and therefore, $[a](R \circ R^{-1}) = [a](R \circ R) = [a]R$. Now

$$[a]\Theta(R) = [a] \left(\bigcup_{n=1}^{\infty} (R \circ R^{-1})^n \right) = \bigcup_{n=1}^{\infty} [a](R \circ R^{-1})^n = \bigcup_{n=1}^{\infty} [a]R = [a]R. \quad \blacksquare$$

3. CONGRUENCES AND CONGRUENCE CLASSES IN BROUWERIAN SEMILATTICES

Definition 3.1. An algebra \mathcal{A} is called an *algebra with 1* if 1 is a distinguished fixed element of the base set of \mathcal{A} . Let \mathcal{A} be an algebra with 1. \mathcal{A} is called *weakly regular* if, for all $\Theta, \Phi \in \text{Con}\mathcal{A}$, $[1]\Theta = [1]\Phi$ implies $\Theta = \Phi$. \mathcal{A} is called *permutable at 1* if, for any $\Theta, \Phi \in \text{Con}\mathcal{A}$, it holds $[1](\Theta \circ \Phi) = [1](\Phi \circ \Theta)$. \mathcal{A} is called *distributive at 1* if, for all $\Theta, \Phi, \Psi \in \text{Con}\mathcal{A}$, it holds $[1](\Theta \vee \Phi) \cap \Psi = [1](\Theta \cap \Psi) \vee (\Phi \cap \Psi)$. \mathcal{A} is called *arithmetic at 1* if it is both permutable at 1 and distributive at 1. A variety \mathcal{V} is called a *variety with 1* if 1 is an equationally definable constant of \mathcal{V} . A variety with 1 is called *weakly regular*, respectively *arithmetic at 1*, if each of its members has the corresponding property.

Proposition 3.1. *A variety \mathcal{V} with 1 is weakly regular if and only if there exist a positive integer n and binary terms t_1, \dots, t_n of \mathcal{V} such that the condition $t_1(x, y) = \dots = t_n(x, y) = 1$ is equivalent to $x = y$, and \mathcal{V} is arithmetic at 1 if and only if there exists a binary term t of \mathcal{V} satisfying $t(x, x) = t(1, x) = 1$ and $t(x, 1) = x$.*

Proof. The first assertion was proved in [2] and the second one in [1]. \blacksquare

Theorem 3.1. *The variety of Brouwerian semilattices is weakly regular and arithmetic at 1.*

Proof. This follows from Proposition 3.1 by taking $n := 2$, $t_1(x, y) := x * y$ and $t_2(x, y) = t(x, y) := y * x$. \blacksquare

Theorem 3.2. For $a, b, c, d \in S$ and $\Theta \in \text{Con}S$, (i)–(ix) hold:

- (i) If $b, c \in [a]\Theta$, then $b \wedge c \in [a]\Theta$.
- (ii) If $b \geq a$, then $b \in [a]\Theta$ if and only if $(b * a) * a \in [a]\Theta$.
- (iii) If $b, c \in [a]\Theta$, then $(b * c) * a \in [a]\Theta$.
- (iv) If $b, c \in [a]\Theta$, then $((b * a) \wedge (c * a)) * a \in [a]\Theta$.
- (v) If $b \Theta c$, then $(b * a) \wedge ((c * a) * a) \in [a]\Theta$.
- (vi) If $b \leq c$, then $b * a \Theta c * a$ if and only if $(b * a) \wedge ((c * a) * a) \in [a]\Theta$.
- (vii) If $b \leq c$ and $(b * a) \wedge ((c * a) * a) \in [a]\Theta$, then
 $((b \wedge d) * a) \wedge (((c \wedge d) * a) * a) \in [a]\Theta$.
- (viii) If $b \leq c$ and $(b * a) \wedge ((c * a) * a) \in [a]\Theta$, then $((b * a) * a) \wedge (c * a) \in [a]\Theta$.
- (ix) If $b, c \wedge (b * a) \in [a]\Theta$, then $c \in [a]\Theta$.

Proof. (i): $b \wedge c \in [a \wedge a]\Theta = [a]\Theta$.

(ii): If $b \in [a]\Theta$, then $(b * a) * a \in [(a * a) * a]\Theta = [a]\Theta$. If, conversely, $(b * a) * a \in [a]\Theta$, then $b = b \wedge 1 = b \wedge (a * a) \in [b \wedge ((b * a) * a)]\Theta = [b \wedge (b * a)]\Theta = [b \wedge a]\Theta = [a]\Theta$.

(iii): $(b * c) * a \in [(a * a) * a]\Theta = [a]\Theta$.

(iv): $((b * a) \wedge (c * a)) * a \in [((a * a) \wedge (a * a)) * a]\Theta = [(a * a) * a]\Theta = [a]\Theta$.

(v): $(b * a) \wedge ((c * a) * a) \in [(b * a) \wedge ((b * a) * a)]\Theta = [(b * a) \wedge a]\Theta = [a]\Theta$.

(vi): If $b * a \Theta c * a$, then $(b * a) \wedge ((c * a) * a) \in [(b * a) \wedge ((b * a) * a)]\Theta = [(b * a) \wedge a]\Theta = [a]\Theta$. Assume, conversely, $(b * a) \wedge ((c * a) * a) \in [a]\Theta$. Then $[(c * a) * a]\Theta \wedge [b * a]\Theta = [a]\Theta$ and, hence, $[b * a]\Theta \leq [(c * a) * a]\Theta * [a]\Theta = [((c * a) * a) * a]\Theta = [c * a]\Theta$. On the other hand, $b \leq c$ implies $c * a \leq b * a$ and, hence, $[c * a]\Theta \leq [b * a]\Theta$. Together it follows $[b * a]\Theta = [c * a]\Theta$ and hence $b * a \Theta c * a$.

(vii): According to (vi), $b * a \Theta c * a$ which implies $((b * a) * a) \wedge d * a \Theta (((c * a) * a) \wedge d) * a$. On the other hand, $b \leq c$ implies $(b * a) * a \leq (c * a) * a$ and, hence, $((b * a) * a) \wedge d \leq ((c * a) * a) \wedge d$. Applying (vi) once more, one obtains

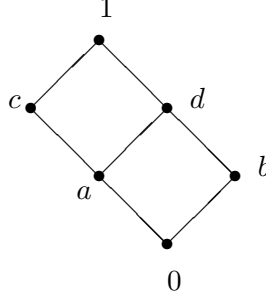
$$((b \wedge d) * a) \wedge (((c \wedge d) * a) * a) = (((b * a) * a) \wedge d) * a \wedge (((c * a) * a) \wedge d) * a \in [a]\Theta.$$

(viii): According to (vi), $b * a \Theta c * a$, whence by (v):

$$((b * a) * a) \wedge (c * a) = ((b * a) * a) \wedge (((c * a) * a) * a) \in [a]\Theta.$$

(ix): $c = c \wedge 1 = c \wedge (a * a) \in [c \wedge (b * a)]\Theta = [a]\Theta$. ■

Remark 3.1. The assumption $b \geq a$ in (ii) cannot be omitted as can be seen from the following example:



If Θ denotes the equivalence relation having the classes $\{0, b\}$, $\{a, d\}$ and $\{c, 1\}$, then $(b * a) * a = c * a = d \in [a]\Theta$ but $b \notin [a]\Theta$. (Note that $b \not\geq a$.)

Corollary 3.1. From (vi), it follows that for $a, b, c \in S$ with $b \leq c$ and, for $\Theta, \Phi \in \text{Con}\mathcal{S}$ with $[a]\Theta = [a]\Phi$, $b * a \Theta c * a$ is equivalent to $b * a \Phi c * a$.

Lemma 3.1. If $a, b \in S$ and $\Theta \in \text{Con}\mathcal{S}$, then $b \in [a]\Theta$ if and only if $a * b, b * a \in [1]\Theta$.

Proof. If $b \in [a]\Theta$, then $a * b, b * a \in [a * a]\Theta = [1]\Theta$. If, conversely, $a * b, b * a \in [1]\Theta$, then $b = b \wedge 1 \in [b \wedge (b * a)]\Theta = [b \wedge a]\Theta = [a \wedge b]\Theta = [a \wedge (a * b)]\Theta = [a \wedge 1]\Theta = [a]\Theta$. ■

Definition 3.2. A subset I of S is called an *ideal* of \mathcal{S} if there exists a congruence Θ on \mathcal{S} with $[1]\Theta = I$. Since the intersection of ideals of \mathcal{S} is again an ideal of \mathcal{S} , there exists a smallest ideal of \mathcal{S} including a given subset M of S . This ideal is called the *ideal of \mathcal{S} generated by M* and it is denoted by $I(M)$. For $a \in S$ and $M \subseteq S$ put $a * M := \{a * x \mid x \in M\}$ and $M * a := \{x * a \mid x \in M\}$.

Lemma 3.2. For $a, b \in S$ and $\Theta \in \text{Con}\mathcal{S}$ the following are equivalent:

- (i) $b \in [a]\Theta$;
- (ii) $a * b, b * a \in (a * ([a]\Theta)) \cup (([a]\Theta) * a)$;
- (iii) $a * b, b * a \in I((a * ([a]\Theta)) \cup (([a]\Theta) * a))$;
- (iv) $a * b, b * a \in [1]\Theta$.

Proof. The implications (i) \Rightarrow (ii) \Rightarrow (iii) are trivial.

(iii) \Rightarrow (iv): $I((a * ([a]\Theta)) \cup (([a]\Theta) * a)) \subseteq [1]\Theta$ follows from Lemma 3.1.

(iv) \Rightarrow (i): follows from Lemma 3.1. ■

Corollary 3.2. *If $a \in S$, $\Theta, \Phi \in \text{Con}\mathcal{S}$ and*

$$I((a * ([a]\Theta)) \cup (([a]\Theta) * a)) = I((a * ([a]\Phi)) \cup (([a]\Phi) * a)),$$

then $[a]\Theta = [a]\Phi$.

Theorem 3.3. *A non-empty subset A of S is a class of some congruence on \mathcal{S} if and only if $a \in A$, $b \in S$ and $a * b, b * a \in I((a * A) \cup (A * a))$ together imply $b \in A$.*

Proof. Assume the condition of the theorem to hold. Let $c \in A$. Then there exists a congruence Ψ on \mathcal{S} with $[1]\Psi = I((c * A) \cup (A * c))$. If $d \in A$, then $c * d, d * c \in I((c * A) \cup (A * c)) = [1]\Psi$ and, hence, $d \in [c]\Psi$ according to Lemma 3.1. If, conversely, $e \in [c]\Psi$, then $c * e, e * c \in [1]\Psi = I((c * A) \cup (A * c))$, because of Lemma 3.1, and, hence $e \in A$ according to the condition of the theorem. This shows $A = [c]\Psi$ and therefore A is a class of some congruence on \mathcal{S} . The rest of the proof follows from Lemma 3.2. ■

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