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MINIMAL FORMATIONS OF UNIVERSAL ALGEBRAS

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Abstract

A class $\mathcal{F}$ of universal algebras is called a *formation* if the following conditions are satisfied: 1) Any homomorphic image of $A \in \mathcal{F}$ is in $\mathcal{F}$; 2) If α_1, α_2 are congruences on A and $A/\alpha_i \in \mathcal{F}, i = 1, 2$, then $A/(\alpha_1 \cap \alpha_2) \in \mathcal{F}$. We prove that any formation generated by a simple algebra with permutable congruences is minimal, and hence any formation containing a simple algebra, with permutable congruences, contains a minimum subformation. This result gives a partial answer to an open problem of Shemetkov and Skiba on formations of finite universal algebras proposed in 1989.

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1. Introduction

Formations of finite groups and formations of finite dimensional Lie algebras have been studied by a number of authors (cf. [3, 2, 4, 6]). Later on, Shemetkov and Skiba [8] have further extended the formation theory of groups to general algebraic systems. A set of universal algebras $\mathfrak{X}$ is said to be a *class of algebras* if $\mathfrak{X}$ contains all isomorphic images of any universal algebra A in $\mathfrak{X}$. A class $\mathfrak{F}$ of universal algebras is called a *formation* if the following conditions are satisfied:

1) Any homomorphic image of $A \in \mathfrak{F}$ is in $\mathfrak{F}$;

2) If α_1, α_2 are congruences on A and $A/\alpha_i \in \mathfrak{F}$, $i = 1, 2$, then $A/(\alpha_a \cap \alpha_2) \in \mathfrak{F}$.

The method of formations has now become an important tool in studying the structure theory of universal algebras.

It was mentioned, e.g., by Mal′cev [5] that every non- identity variety of universal algebras contains a minimal variety. In 1984, Shemetkov [7] proved that every variety of universal algebras is a formation. In this connection, he and Skiba proposed the following interesting question in their monograph [8], (see Problem 5.15):

Is it true that any formation of finite universal algebras has a minimal subformation?

In this paper, we give an affirmative answer to the above problem for finite universal algebras with permutable congruences, that is, our result partly answers the problem. What we proved in our paper is that if the universal algebras have permutable congruences and satisfy the maximum condition for congruences then the formation of such algebras always contains minimal subformations. Note that every finite universal algebra always satisfies the maximum condition for congruences. For notations and definitions not given in this paper, the reader is referred to the monograph of Shemetkov and Skiba [8].

2. Existence of minimal formations

Throughout this paper, all universal algebras are algebras with permutable congruences and satisfying the maximum condition for congruences. By an $\mathfrak{X}$-algebra A, we mean $A \in \mathfrak{X}$, where $\mathfrak{X}$ is a set of universal algebras.

A congruence φ on an universal algebra A is called *completely decomposable* if for any congruence $\pi \subseteq \varphi$, there exists a congruence ψ on A such that $\varphi = \pi\psi$ and $\pi \cap \psi = \triangle$, the identity congruence.

An universal algebra A is called *simple* if there is no proper congruence φ on A, that is, if $\triangle \subseteq \varphi \subseteq A^2$, then $\varphi = \triangle$ or A^2, where A^2 is the global congruence $A \times A$ on A.

The following are the notations which will be used in this paper.

1. $soc(A)$: the congruence generated by all minimal congruences on the universal algebra A.
2. $H(\mathfrak{X})$: the class of all homomorphic images of all $\mathfrak{X}$-universal algebras.
3. $R_0(\mathfrak{X})$: the class of all finite subdirect product of $\mathfrak{X}$-universal algebras.
4. $form\mathfrak{X}$: the intersection of all formations containing $\mathfrak{X}$. In particular, if $\mathfrak{X} = \{A\}$, then $form\{A\}$ is denoted by $formA$ and is called the formation generated by A.

The following lemmas for universal algebras with permutable congruences are crucial in solving the proposed problem.

Lemma 1 ([8], Lemma 3.2). *Let $\mathfrak{X}$ be a set of universal algebras. Then $form\mathfrak{X} = HR_0(\mathfrak{X})$.*

Lemma 2 ([8], Lemma 3.16). *Let H be a subdirect product of simple universal algebras $A_1, A_2, \cdots, A_t$. Then there are universal algebras $B_1, B_2, \cdots, B_r$ ($r \leqslant t$) such that $H = B_1 \times B_2 \times \cdots \times B_r$ and for any $i \in \{1, 2, \cdots, r\}$, B_i is isomorphic to one of the universal algebras $A_1, A_2, \cdots, A_t$.*

Lemma 3 ([8], Lemma 3.14). *Let $\pi \neq \triangle$ be a congruence on an universal algebra A. Then π is completely decomposable if and only if π is generated by a set of minimal congruences on A.*

Lemma 4 ([1] p. 303). *If φ_1, φ_2 are congruences on an universal algebra A such that $\varphi_1\varphi_2 = A^2$ and $\varphi_1 \cap \varphi_2 = \triangle$, then $A \cong A/\varphi_1 \times A/\varphi_2$.*

By using the above lemmas, we are now able to formulate a theorem of minimal subformations. Of course, this theorem only holds for algebras with permutable congruences otherwise we cannot use the results in the above lemmas.

Theorem 5. *A non-identity formation $\mathfrak{F}$ of universal algebras with permutable congruences satisfying the maximum condition for congruences always contains a minimal subformation.*

Proof. Let $\mathcal{F}$ be a non-identity formation of universal algebras. Then there exists a non-trivial universal algebra in $\mathcal{F}$. Since we assume that the congruences on every universal algebra in $\mathcal{F}$ satisfy the maximum condition, every collection of congruences on any algebra in $\mathcal{F}$ must contain a maximal member. In particular, this implies that there exists a simple algebra A in $\mathcal{F}$. We now prove that the formation $formA$ generated by this simple algebra A does not contain a proper non-identity subformation.
Suppose, on the contrary, that there is a non-identity formation $\mathcal{M} \subseteq formA$. Then we can let B be a non-trivial universal algebra in $\mathcal{M}$. Since B is in $\mathcal{M} \subseteq formA = HR_0(A)$ by Lemma 1, we have $B \simeq H/\pi$, where H is a subdirect product of some copies of A and π is a congruence on H. Now, by Lemma 2, $H = A_1 \times A_2 \times \cdots \times A_t$, where $A_i \simeq A$ for all $i \in \{1, 2, \cdots, t\}$. Hence, by the definition of socles, we see that $soc(H) = H^2 = H \times H$, i.e., H^2 is generated by all minimal congruences on H. By invoking Lemma 3, H^2 is completely decomposable. This implies that there exists a congruence φ on H such that

$$H^2 = \pi\varphi \text{ and } \pi \cap \varphi = \triangle.$$

Furthermore, by Lemma 4, we have $H \simeq H/\pi \times H/\varphi$. Since $H = A_1 \times A_2 \times \cdots \times A_t$ and $B \simeq H/\pi$, we conclude that $B \simeq H/\pi \simeq A_{i_1} \times \cdots \times A_{i_s}$, for some $A_{ij} \in \{A_1, A_2, \cdots, A_t\}$. Consequently, we have $A \simeq A_{i_1} \simeq B/\psi$, where ψ is the kernel congruence determined by the projection of B to A_{i_1}. This shows that $A \in formB \subseteq \mathcal{M}$, and hence $\mathcal{M} = formA$. The theorem is proved.

Since we have already seen in the proof of Theorem 5 that any formation generated by a simple algebra is minimal, we can reformulate Theorem 5 in the following nicer form.

Theorem 6. *Any formation containing a simple algebra with permutable congruences always contains a minimal subformation.*

We remark here again that the problem proposed by Shemetkov and Skiba [8] is now answered by our Theorem 5 if the algebras concerned are algebras with permutable congruences, since the maximal condition of congruences trivially holds for finite algebras. It is noted that for groups, Ω-groups, rings, linear algebras, multirings and m-groups etc., the congruences on these algebras are always permutable. Therefore, it is natural for us to consider algebras with permutable congruences. However, if the conditions "permutable congruences" is removed from Theorem 5, then we will be unable to make use of our Lemmas to prove Theorem 5.

In closing this paper, we propose the following question.

Will the problem of Shemetkov and Skiba still be true for the formation of universal algebras with maximal condition of congruences but without the condition of permutable congruences?

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