

ON THE STRUCTURE OF
HALFDIAGONAL-HALFTERMINAL-SYMMETRIC
CATEGORIES WITH DIAGONAL INVERSIONS

HANS-JÜRGEN VOGEL

University of Potsdam Institute of Mathematics
PF 60 15 53 D-14415 Potsdam, Germany
e-mail: vogel@rz.uni-potsdam.de

Dedicated to Hans-Jürgen Hoehnke on the occasion of his 75th birthday.

Abstract

The category of all binary relations between arbitrary sets turns out to be a certain symmetric monoidal category \mathbf{Rel} with an additional structure characterized by a family $d = (d_A : A \rightarrow A \otimes A \mid A \in |\mathbf{Rel}|)$ of diagonal morphisms, a family $t = (t_A : A \rightarrow I \mid A \in |\mathbf{Rel}|)$ of terminal morphisms, and a family $\nabla = (\nabla_A : A \otimes A \rightarrow A \mid A \in |\mathbf{Rel}|)$ of diagonal inversions having certain properties. Using these properties in [11] was given a system of axioms which characterizes the abstract concept of a halfdiagonal-halfterminal-symmetric monoidal category with diagonal inversions (*hdht* ∇ -category). Besides of certain identities this system of axioms contains two identical implications. In this paper it is shown that there is an equivalent characterizing system of axioms for *hdht* ∇ -categories consisting of identities only. Therefore, the class of all small *hdht* ∇ -symmetric categories (interpreted as heterogeneous algebras of a certain type) forms a variety and hence there are free theories for relational structures.

Keywords: halfdiagonal-halfterminal-symmetric category, diagonal inversion, partial order relation, subidentity, equation.

2000 AMS Subject Classification: 18D10, 18B10, 18D20, 08A05, 08A02.

1. Defining conditions

Let K^\bullet be any symmetric monoidal category in the sense of Eilenberg-Kelly ([2]) with the object class $|K|$, the morphism class K , the distinguished object I , the bifunctor $\otimes : K \times K \rightarrow K$, and the families a, r, l, s of isomorphisms of K such that the following axioms are valid for all objects and all morphisms of K . By $K[A, B]$ we denote the set of all morphisms $\rho \in K$ with the domain (source) $\text{dom } \rho = A$ and the codomain (target) $\text{codom } \rho = B$.

Bifunctor properties:

- (F1) $\text{dom } (\rho \otimes \rho') = \text{dom } \rho \otimes \text{dom } \rho'$,
- (F2) $\text{codom } (\rho \otimes \rho') = \text{codom } \rho \otimes \text{codom } \rho'$,
- (F3) $1_{A \otimes B} = 1_A \otimes 1_B$,
- (F4) $(\rho \otimes \rho')(\sigma \otimes \sigma') = \rho\sigma \otimes \rho'\sigma'$.

Conditions of monoidality:

- (M1) $a_{A,B,C} \otimes a_{A \otimes B, C, D} = (1_A \otimes a_{A,B,C}) a_{A, B \otimes C, D} (a_{A,B,C} \otimes 1_D)$,
- (M2) $a_{A,I,B} (r_A \otimes 1_B) = 1_A \otimes l_B$,
- (M3) $a_{A,B,C} s_{A \otimes B, C} a_{C,A,B} = (1_A \otimes s_{B,C}) a_{A,C,B} (s_{A,C} \otimes 1_B)$,
- (M4) $s_{A,B} s_{B,A} = 1_{A \otimes B}$,
- (M5) $s_{A,I} l_A = r_A$,
- (M6) $a_{A,B,C} ((\rho \otimes \sigma) \otimes \tau) = (\rho \otimes (\sigma \otimes \tau)) a_{A',B',C'}$,
- (M7) $r_A \rho = (\rho \otimes 1_I) r_{A'}$,
- (M8) $s_{A,B} (\sigma \otimes \rho) = (\rho \otimes \sigma) s_{A',B'}$.

Remark that the validity of an equation containing morphism compositions includes that they are defined on both sides.

An immediate consequence of the conditions above is the validity of

- (M9) $\forall A, B \in |K| (a_{I,A,B} (l_A \otimes 1_B) = l_{A \otimes B})$,
- (M10) $\forall A, B \in |K| (a_{A,B,I} r_{A \otimes B} = 1_A \otimes r_B)$,
- (M11) $r_I = l_I$,
- (M12) $s_{I,I} = 1_{I \otimes I}$,

$$(M13) \quad \forall A \in |K| \ (s_{I,A}r_A = l_A),$$

$$(M14) \quad \forall A \in |K| \ (l_A\rho = (1_I \otimes \rho)l_{A'}).$$

Using the denotation

$$b_{A,B,C,D} := a_{A \otimes B, C, D}(a_{A, B, C}^{-1}(1_A \otimes s_{B, C})a_{A, C, B} \otimes 1_D)a_{A \otimes C, B, D}^{-1}$$

one obtains the following properties for all objects $A, A', B, B', C, C', D, D'$ of K and all morphisms $\rho \in K[A, A']$, $\sigma \in K[B, B']$, $\lambda \in K[C, C']$, $\mu \in K[D, D']$:

$$(M15) \quad b_{A,B,C,D}((\rho \otimes \sigma) \otimes (\lambda \otimes \mu)) = ((\rho \otimes \lambda) \otimes (\sigma \otimes \mu))b_{A',B',C',D'},$$

$$(M16) \quad b_{A,I,I,B} = 1_{A \otimes I} \otimes 1_{I \otimes B},$$

$$(M17) \quad b_{A,B,C,D}b_{A,C,B,D} = 1_{A \times B} \otimes 1_{C \otimes D},$$

$$(M18) \quad b_{A,B,C,D}(s_{A,C} \otimes s_{B,D}) = s_{A \otimes B, C \otimes D}b_{C,D,A,B}.$$

Obviously, all morphisms $b_{A,B,C,D}$ are isomorphisms in the category K^\bullet .

Definition 1.1 ([1]). A *diagonal-terminal-symmetric category* (shortly *dtS-category*) $\underline{K} = (K^\bullet, d, t)$ is defined as a symmetric monoidal category endowed with morphism families

$$d = (d_A : A \rightarrow A \otimes A \mid A \in |K|) \quad \text{and} \quad t = (t_A : A \rightarrow I \mid A \in |K|)$$

satisfying the following conditions for all objects $A, B, A' \in |K|$ and all morphisms $\rho \in K[A, A']$.

Diagonality:

$$(D1) \quad d_A(d_A \otimes 1_A) = d_A(1_A \otimes d_A)a_{A,A,A},$$

$$(D2) \quad d_A s_{A,A} = d_A,$$

$$(D3) \quad d_{A \otimes B} = (d_A \otimes d_B)b_{A,A,B,B},$$

$$(D4) \quad d_A(\rho \otimes \rho) = \rho d_{A'}.$$

Terminality:

$$(T1) \quad d_A(1_A \otimes t_A)r_A = 1_A,$$

$$(T2) \quad t_I = 1_I,$$

$$(T3) \quad \rho t_{A'} = t_A.$$

Let A, A', B be arbitrary objects in K and let $\rho \in K[A, A']$ be any morphism in K . Then the properties

$$\begin{aligned}
(\text{D5}) \quad & d_A(d_A \otimes d_A) = d_A d_{A \otimes A}, \\
(\text{D6}) \quad & d_A(d_A \otimes d_A) = d_A(d_A \otimes d_A) b_{A, A, A, A}, \\
(\text{D7}) \quad & t_A d_I = d_A(t_A \otimes t_A), \\
(\text{D9}) \quad & \rho d_{A'} d_{A' \otimes A'} = d_A(\rho d_{A'} \otimes d_A(\rho \otimes \rho)), \\
(\text{T4}) \quad & d_A(t_A \otimes 1_A) l_A = 1_A, \\
(\text{T5}) \quad & d_{A \otimes B}((1_A \otimes t_B) r_A \otimes (t_A \otimes 1_B) l_B) = 1_{A \otimes B}, \\
(\text{T6}) \quad & t_{A \otimes B} = (t_A \otimes t_B) t_{I \otimes I}, \\
(\text{T7}) \quad & r_I = t_{I \otimes I}, \\
(\text{T8}) \quad & d_A t_{A \otimes A} = t_A, \\
(\text{T9}) \quad & \rho t_{A'} d_I = d_A(\rho t_{A'} \otimes t_A)
\end{aligned}$$

are consequences of the conditions above ([1]).

The category *Set* of all total functions between arbitrary sets is a model of a *dts*-category by

$$I := \{\emptyset\}, \quad A \otimes B := \{\langle a, b \rangle \mid a \in A \wedge b \in B\},$$

$$\rho \in \text{Set}[A, B] := \rho = \{(a, b) \mid a \in A \wedge b = \rho(a) \in B\},$$

$$\forall a \in A \exists!! b \in B (b = \rho(a)),$$

$$\rho \in \text{Set}[A, B], \sigma \in \text{Set}[B, C] \Rightarrow \rho \circ \sigma := \{(a, c) \mid a \in A \wedge c = \sigma(\rho(a))\},$$

$$(a, c) \in \rho \circ \sigma \Leftrightarrow \exists b \in B ((a, b) \in \rho \wedge (b, c) \in \sigma),$$

$$\rho \in \text{Set}[A, B], \rho' \in \text{Set}[A', B'] \Rightarrow \rho \otimes \rho' := \{(\langle a, a' \rangle, \langle \rho(a), \rho'(a') \rangle) \mid a \in A, a' \in A'\},$$

$$a_{A, B, C} := \{(\langle a, \langle b, c \rangle \rangle, \langle \langle a, b \rangle, c \rangle) \mid a \in A, b \in B, c \in C\},$$

$$s_{A, B} := \{(\langle a, b \rangle, \langle b, a \rangle) \mid a \in A, b \in B\},$$

$$r_A := \{(\langle a, \emptyset \rangle, a) \mid a \in A\},$$

$$l_A := \{(\langle \emptyset, a \rangle, a) \mid a \in A\},$$

$$d_A := \{(a, \langle a, a \rangle) \mid a \in A\},$$

$$t_A := \{(a, \emptyset) \mid a \in A\}.$$

Remark that I is a *terminal object* in any *dts*-category \underline{K} and $(A \otimes B; p_1^{A,B}, p_2^{A,B})$ forms a *categorical product* of the objects A, B in the category K , where $p_1^{A,B} := (1_A \otimes t_B)r_A$ and $p_2^{A,B} := (t_A \otimes 1_B)l_B$.

Moreover, $d_A(\rho \otimes \sigma) = \rho d_B$ is equivalent to $\rho = \sigma$ for all $A, B \in |K|$ and all $\rho, \sigma \in K[A, B]$ because of

$$\begin{aligned} \sigma &= \sigma d_B p_2^{B,B} = d_A(\sigma t_B \otimes \sigma)l_B = d_A(t_A \otimes \sigma)l_B \\ &= d_A(\rho t_B \otimes \sigma)l_B = d_A(\rho \otimes \sigma)p_2^{B,B} = \rho d_B p_2^{B,B} = \rho. \end{aligned}$$

The morphisms $p_1^{A,B}$ and $p_2^{A,B}$ are called *canonical projections* in the category K .

Conditions (D9) and (T9) are equivalent to

$$\rho d_{A'} = d_A(\rho d_{A'} \otimes d_A(\rho \otimes \rho))p_2^{A',A'} \quad \text{and} \quad \rho t_{A'} = d_A(\rho t_{A'} \otimes t_A)p_2^{I,I}, \text{ respectively.}$$

Definition 1.2. Let K^\bullet be again a symmetric monoidal category endowed with morphism families d and t as above. Then $\underline{K} = (K^\bullet, d, t)$ is called *halfdiagonal-terminal-symmetric category* (shortly *hdts-category*), if the conditions

$$(D1), (D2), (D3), (D5), (D7), (T1), (T2), (T3)$$

hold identically.

As above, the identities (T4), (T5), (T6), (T7), (T8), (T9) follow from the defining conditions in an *hdts*-category.

Definition 1.3. A *diagonal-halfterminal-symmetric category* (shortly *dhts*-category) ([3], [7], [10]) is defined as a sequence $\underline{K} := (K^\bullet; d, t, O, o)$ such that K^\bullet is again a symmetric monoidal category, d and t are families as above, O is a distinguished zero-object of K^\bullet , $o : I \rightarrow O$ is a distinguished morphism of K^\bullet , and the following equations are fulfilled for all objects $A, B, A', B' \in |K|$ and all morphisms $\rho \in K[A, A']$, $\sigma \in K[B, B']$, $\lambda \in K[A, O]$, $\kappa \in K[O, A]$:

$$(D4), (T1), (T4), (T5), (T6), \text{ and}$$

$$(o1) \quad t_A o = \lambda,$$

$$(o2) \quad (1_A \otimes t_O)r_A = \kappa,$$

$$(O1) \quad A \otimes O = O \otimes A = O.$$

Remark that the conditions

(D1), (D2), (D3), (D5), (D6), (D7), (D9), (T2), (T7), (T8), (T9),
and

$$(B1) \quad b_{A,B,C,D}(1_{A \otimes C} \otimes t_{B \otimes D})r_{A \otimes C} = (1_A \otimes t_B)r_A \otimes (1_C \otimes t_D)r_C,$$

$$(B2) \quad b_{A,B,C,D}(t_{A \otimes C} \otimes 1_{B \otimes D})l_{B \otimes D} = (t_A \otimes 1_B)l_B \otimes (t_C \otimes 1_D)l_D$$

are consequences of the other conditions ([3], [7], [10]).

Formulas (o1), (o2), and (O1) explain that the morphism sets $K[A, O]$ and $K[O, A]$ both consist of exactly one element $o_{A,O}$ and $o_{O,A}$, respectively, and O is a zero object in K . In any *dhts*-category there is a so-called *zero-morphism* $o_{A,B}$ to each pair of objects $A, B \in |K|$ with the properties

$$(o3) \quad \forall \rho \in K[A, A'], \sigma \in K[B, B'] \quad (\rho o_{A,B} = o_{A',B} \wedge o_{A,B} \sigma = o_{A,B'}),$$

$$(o4) \quad \forall \xi, \eta \in K \quad (o_{A,B} \otimes \xi = o_{A,B} = \eta \otimes o_{A,B}),$$

$$(o5) \quad o_{O,A} = (1_A \otimes t_O)r_A = (t_O \otimes 1_A)l_A.$$

The category *Par* of all partial functions between arbitrary sets is a model of a *dhts*-category by the same fixations as above and $O = \emptyset$ (the empty set) and $o : I \rightarrow O$, $o_{A,O} : A \rightarrow O$, $o_{O,A} : O \rightarrow A$, $o_{A,B} : A \rightarrow B$ as the empty functions. The morphisms are given by

$$\begin{aligned} \rho \in K[A, B] \quad &:\Leftrightarrow \quad \rho = \{(a, \rho(a)) \mid a \in D(\rho) \wedge \rho(a) \in B\}, \\ &\forall a \in D(\rho) \subseteq A \exists!! b \in B \quad (b = \rho(a)). \end{aligned}$$

The following fact is of importance for the consideration of *dhts*-categories.

Lemma 1.4. *Let \underline{K} be a symmetric monoidal category endowed with morphism families d and t as above which fulfil conditions (D4), (T1) and (T6). Then conditions (T4) and (T5) are consequences of the validity of (D2) and (D3) in \underline{K} .*

Proof. Using (T1) and (D2) one obtains (T4) as follows:

$$1_A = d_A(1_A \otimes t_A)r_A = d_A s_{A,A}(1_A \otimes t_A)r_A = d_A(t_A \otimes 1_A)s_{I,A}r_A = d_A(t_A \otimes 1_A)l_A.$$

The calculation

$$\begin{aligned}
& d_{A \otimes B}((1_A \otimes t_B)r_A \otimes (t_A \otimes 1_B)l_B) \\
&= (d_A \otimes d_B)b_{A,A,B,B}((1_A \otimes t_B)r_A \otimes (t_A \otimes 1_B)l_B) && ((D3)) \\
&= (d_A(1_A \otimes t_A) \otimes d_B(t_B \otimes 1_B))b_{A,I,I,B}(r_A \otimes l_B) && ((M15)) \\
&= (d_A(1_A \otimes t_A) \otimes d_B(t_B \otimes 1_B))(1_{A \otimes I} \otimes 1_{I \otimes B})(r_A \otimes l_B) && ((M16)) \\
&= (d_A(1_A \otimes t_A) \otimes d_B(t_B \otimes 1_B))(r_A \otimes l_B) && ((F3)) \\
&= (d_A(1_A \otimes t_A)r_A \otimes d_B(t_B \otimes 1_B)l_B) && ((F4)) \\
&= 1_A \otimes 1_B && ((T1), (T4))
\end{aligned}$$

shows the validity of (T5). ■

Let \underline{K} be an arbitrary *dhts*-category. Then all morphisms $\rho \in K[A, A']$, $A, A' \in |K|$, fulfilling $\rho t_{A'} = t_A$, form a subcategory \underline{M}^K of \underline{K} which is even a *dtc*-category. Denoting by \underline{M}_K the smallest *dtc*-subcategory of \underline{M}^K containing all morphisms of the families a, r, l, s, d, t one has

$$\underline{M}_K \subseteq \underline{\text{Iso}}(K) \subseteq \underline{\text{Cor}}(K) \subseteq \underline{M}^K,$$

where $\underline{\text{Iso}}(K)$ ($\underline{\text{Cor}}(K)$) is a *dtc*-subcategory of \underline{M}^K generated by all isomorphisms (coretractions) of K together with all terminal morphisms of K , since all coretractions and all terminal morphisms fulfil the condition (T3) (see [7], [10]).

The object $I \in |K|$ is a terminal object in the subcategories $\underline{M}_K, \underline{\text{Iso}}(K), \underline{\text{Cor}}(K)$, and \underline{M}^K but not in the whole category \underline{K} . Morphisms of the kind $p_1^{A,B} = (1_A \otimes t_B)r_A$ and $p_2^{A,B} = (t_A \otimes 1_B)l_B$ are called *canonical projections* again and $(A \otimes B; p_1^{A,B}, p_2^{A,B})$ is a *categorical product* of A and B in \underline{M}^K , but in general not in the whole category.

Schreckenberger had proved ([7]) that

$$\rho \leq \sigma \Leftrightarrow d_A(\rho \otimes \sigma) = \rho d_{A'} \quad (\rho, \sigma \in K[A, A'])$$

defines a partial order relation which is stable under composition and \otimes -operation. Moreover, the following are equivalent:

- (i) $d_A(\rho \otimes \sigma) = \rho d_{A'}$,
- (ii) $d_A(\rho \otimes \sigma)p_2^{A',A'} = \rho$,
- (iii) $d_A(\sigma \otimes \rho)p_1^{A',A'} = \rho$.

Hoehnke had shown ([3]) the validity of the identical implication

$$\rho = d_A(\rho \otimes \sigma)p_2^{A',A'} \Rightarrow \rho = d_A(\rho \otimes \sigma)p_1^{A',A'}.$$

The relation \leq in the *dhts*-category \underline{Par} describes exactly the usual inclusion \subseteq .

Morphisms $e_A \in K[A, A]$ of any *dhts*-category \underline{K} fulfilling $e_A \leq 1_A$ for any $A \in |K|$ are called *subidentities* ([7]). Especially, for each $\rho \in K[A, B]$, the morphism

$$\alpha(\rho) := d_A(\rho \otimes 1_A)p_2^{B,A} (= d_A(1_A \otimes \rho)p_1^{A,B})$$

is a subidentity of $A \in |K|$, since

$$\begin{aligned} d_A(d_A(\rho \otimes 1_A)p_2^{B,A} \otimes 1_A)p_2^{A,A} &= d_A(\rho \otimes d_A(1_A \otimes 1_A))a_{B,A,A}(p_2^{B,A} \otimes 1_A)p_2^{A,A} \\ &= d_A(\rho \otimes d_A)(1_B \otimes p_2^{A,A})p_2^{B,A} \\ &= d_A(\rho \otimes d_A p_2^{A,A})p_2^{B,A} = d_A(\rho \otimes 1_A)p_2^{B,A}. \end{aligned}$$

Important properties of subidentities are described in [7], [13], [15].

Definition 1.5. A *diagonal-halfterminal-symmetric category with diagonal inversion* ∇ (shortly *dht* ∇ -*category*, [10]) is, by definition, a sequence $\underline{K} := (K^\bullet; d, t, \nabla, O, o)$ such that $(K^\bullet; d, t, O, o)$ is a *dhts*-category endowed with a morphism family $\nabla = (\nabla_A \mid A \in |K|)$ satisfying the following for all $A \in |K|$:

- ($\nabla 1$) $d_A \nabla_A = 1_A$,
- ($\nabla 2$) $\nabla_A d_A d_{A \otimes A} = d_{A \otimes A}(\nabla_A d_A \otimes 1_{A \otimes A})$.

The category \underline{Par} is also a model of a *dht* ∇ -*category*, where

$$\nabla_A := \{((a, a), a) \mid a \in A\}, \quad A \in |\underline{Par}|.$$

The properties

$$\begin{aligned}
(\text{D8}) \quad & \nabla_A d_A = d_{A \otimes A}(\nabla_A \otimes \nabla_A), \\
(\text{D9}') \quad & \rho d_{A'} = d_A(\rho d_{A'} \otimes d_A(\rho \otimes \rho)) \nabla_{A' \otimes A'}, \\
(\text{T9}') \quad & \rho t_{A'} = d_A(\rho t_{A'} \otimes t_A) \nabla_I, \\
(\nabla 3) \quad & a_{A,A,A}(\nabla_A \otimes 1_A) \nabla_A = (1_A \otimes \nabla_A) \nabla_A, \\
(\nabla 4) \quad & s_{A,A} \nabla_A = \nabla_A, \\
(\nabla 5) \quad & \nabla_{A \otimes B} = b_{A,B,A,B}(\nabla_A \otimes \nabla_B), \\
(\nabla 6) \quad & \nabla_A d_A = (d_A \otimes 1_A) a_{A,A,A}^{-1}(1_A \otimes \nabla_A), \\
(\nabla 7) \quad & \nabla_A d_A = (1_A \otimes d_A) a_{A,A,A}(\nabla_A \otimes 1_A), \\
(\nabla 8) \quad & \nabla_A d_A = (d_A \otimes d_A) \nabla_{A \otimes A}, \\
(\nabla 9) \quad & \nabla_A \rho d_{A'} = d_{A \otimes A}(\nabla_A \rho \otimes (\rho \otimes \rho) \nabla_{A'}), \\
(\nabla 9') \quad & \nabla_A \rho = d_{A \otimes A}(\nabla_A \rho \otimes (\rho \otimes \rho) \nabla_{A'}) \nabla_{A'}, \\
(\nabla 10) \quad & \nabla_{A \otimes A} \nabla_A = (\nabla_A \otimes \nabla_A) \nabla_A, \\
(\text{D}\nabla) \quad & \rho = d_A(\rho \otimes \rho) \nabla_{A'}
\end{aligned}$$

follow from the axioms and the other properties of a $dht\nabla s$ -category for all $A, A', B \in |K|$ and all $\rho \in K[A, A']$ (see [13]).

By the definition of the partial order relation, (T9) is equivalent to $\rho t_{A'} \leq t_A$, ($\nabla 2$) is equivalent to $\nabla_A d_A \leq 1_{A^2}$, and ($\nabla 9$) is equivalent to $\nabla_A \rho \leq (\rho \otimes \rho) \nabla_{A'}$ for $\rho \in K[A, A']$.

Moreover, one has the following important property in any $dht\nabla s$ -category \underline{K} ([11]):

$$(\text{P}\nabla) \quad \forall A, A' \in |K| \quad \forall \rho, \sigma \in K[A, A'] \quad (d_A(\rho \otimes \sigma) p_2^{A', A'} = \rho \Leftrightarrow d_A(\rho \otimes \sigma) \nabla_{A'} = \rho).$$

In any $dht\nabla s$ -category, conditions (D9), (T9), and ($\nabla 9$) result in (D9'), (T9'), and ($\nabla 9'$), respectively.

2. $hdht\nabla s$ -categories

Definition 2.1 ([10]). A sequence $\underline{K} = (K^\bullet; d, t, \nabla, o)$ is called *halfdiagonal-halfterminal-symmetric monoidal category with diagonal inversion* ∇ (shortly *hdht* ∇s -category), iff K^\bullet is a symmetric monoidal category as above,

$(d_A : A \rightarrow A \otimes A \mid A \in |K|)$, $(t_A : A \rightarrow I \mid A \in |K|)$, $(\nabla_A : A \otimes A \rightarrow A \mid A \in |K|)$ are families of morphisms of K , and $o : I \rightarrow O$ ($I \neq O \in |K|$) is a distinguished morphism of K such that for all objects and all morphisms of the underlying category K the conditions

$$(D1), (D2), (D3), (D5), (D7), (D8),$$

$$(T1), (T2), (T6), (T9'),$$

$$(\nabla 1), (\nabla 2), (\nabla 3), (\nabla 4), (\nabla 5), (D\nabla),$$

$$(o1), (o2), (O1),$$

and

$$(*1) \quad d_A(\rho \otimes \rho') \nabla_B d_B(\sigma \otimes \sigma') \nabla_C \\ = d_A(d_A(\rho \otimes \rho') \nabla_B d_B(\sigma \otimes \sigma') \nabla_C \otimes d_A(\rho \otimes \rho') \nabla_C) \nabla_C$$

are fulfilled.

The system of axioms given in this definition is free of contradictions, because the category Rel of all binary relations between sets is a model of it, i.e. Rel fulfils all the axioms of an *hdht* ∇ -category, where $|Rel|$ is the class of all sets, the morphisms are characterized by

$$\rho \in Rel[A, A'] : \Leftrightarrow \rho = \{(a, a') \mid a \in D(\rho) \subseteq A \wedge a' \in W(\rho) \subseteq A' \wedge H(a, a')\},$$

where $H(x, y)$ is a sentence form in two variables, the distinguished objects are $I = \{\emptyset\}$ and $O = \emptyset$, the operation \otimes for objects is given as in *Set*, the composition and the \otimes -operation of morphisms are described by

$$\rho \in Rel[A, B], \sigma \in Rel[B, C] \Rightarrow \rho \circ \sigma = \{(a, c) \mid \exists b \in B ((a, b) \in \rho \wedge (b, c) \in \sigma)\},$$

$$\rho \in Rel[A, B], \rho' \in Rel[A', B'] \Rightarrow \rho \otimes \rho' = \{(\langle a, a' \rangle, \langle b, b' \rangle) \mid (a, b) \in \rho \wedge (a', b') \in \rho'\},$$

and the morphisms of the families $a, r, l, s, b, d, t, \nabla, (0_{A,B} \mid A, B \in |Rel|)$ are as in *Par*.

Lemma 2.2. *The relation \leq defined by*

$$\rho \leq \sigma : \Leftrightarrow d_A(\rho \otimes \sigma) \nabla_B = \rho$$

is a partial order relation in any hdht ∇ -*symmetric category which is compatible with composition and* \otimes -*operation for morphisms. Moreover, the greatest*

lower bound of two morphisms $\lambda, \mu \in K[A, B]$ with respect to the canonical order relation \leq is given by

$$d_A(\lambda \otimes \mu)\nabla_B = \inf\{\lambda, \mu\}.$$

Proof. Condition $(D\nabla)$ shows the reflexivity of \leq . The relation is anti-symmetric because of

$$\begin{aligned} \rho \leq \sigma \wedge \sigma \leq \rho &\Rightarrow \sigma = d_A(\sigma \otimes \rho)\nabla_B \\ &= d_A s_{A,A}(\sigma \otimes \rho)\nabla_B && ((D2)) \\ &= d_A(\rho \otimes \sigma) s_{B,B}\nabla_B && ((M8)) \\ &= d_A(\rho \otimes \sigma)\nabla_B && ((\nabla4)) \\ &= \rho. \end{aligned}$$

The implication

$$\begin{aligned} \rho \leq \sigma \wedge \sigma \leq \tau &\Rightarrow \rho = d_A(\rho \otimes \sigma)\nabla_B \\ &= d_A(\rho \otimes d_A(\sigma \otimes \tau)\nabla_B)\nabla_B \\ &= d_A(1_A \otimes d_A)(\rho \otimes (\sigma \otimes \tau))(1_B \otimes \nabla_B)\nabla_B \\ &= d_A(d_A \otimes 1_A)((\rho \otimes \sigma) \otimes \tau) a_{B,B,B}^{-1}(1_B \otimes \nabla_B)\nabla_B && ((M6), (D1)) \\ &= d_A(d_A(\rho \otimes \sigma) \otimes \tau)(\nabla_B \otimes 1_B)\nabla_B && ((\nabla3)) \\ &= d_A(d_A(\rho \otimes \sigma)\nabla_B \otimes \tau)\nabla_B \\ &= d_A(\rho \otimes \tau)\nabla_B \\ &\Rightarrow \rho \leq \tau \end{aligned}$$

yields the transitivity of the relation \leq .

Now suppose $\rho \leq \sigma$, $\lambda \leq \mu$, and $\text{cod } \rho = \text{dom } \lambda$. Then $\rho\lambda \leq \sigma\mu$ follows via the definition of \leq by condition $(*1)$:

$$\begin{aligned}
\rho \leq \sigma \wedge \lambda \leq \mu &\Rightarrow d_A(\rho \otimes \sigma) \nabla_B = \rho \wedge d_B(\lambda \otimes \mu) \nabla_C = \lambda \\
&\Rightarrow \rho \lambda = d_A(\rho \otimes \sigma) \nabla_B d_B(\lambda \otimes \mu) \nabla_C \\
&= d_A(d_A(\rho \otimes \sigma) \nabla_B d_B(\lambda \otimes \mu) \nabla_C \otimes d_A(\rho \lambda \otimes \sigma \mu) \nabla_C) \nabla_C \\
&= d_A(\rho \lambda \otimes d_A(\rho \lambda \otimes \sigma \mu) \nabla_C) \nabla_C \\
&= d_A(d_A(\rho \lambda \otimes \rho \lambda) \otimes \sigma \mu) a_{C,C,C}^{-1} (1_C \otimes \nabla_C) \nabla_C \\
&= d_A(\rho \lambda \otimes \rho \lambda) \nabla_C \otimes \sigma \mu \nabla_C \\
&= d_A(\rho \lambda \otimes \sigma \mu) \nabla_C \\
&\Rightarrow \rho \lambda \leq \sigma \mu.
\end{aligned}$$

For morphisms $\rho \leq \sigma \in K[A, B]$ and $\rho' \leq \sigma' \in K[A', B']$ one obtains

$$\rho = d_A(\rho \otimes \sigma) \nabla_B \quad \text{and} \quad \rho' = d_{A'}(\rho' \otimes \sigma') \nabla_{B'},$$

hence

$$\begin{aligned}
\rho \otimes \rho' &= d_A(\rho \otimes \sigma) \nabla_B \otimes d_{A'}(\rho' \otimes \sigma') \nabla_{B'} \\
&= (d_A \otimes d_{A'})((\rho \otimes \sigma) \otimes (\rho' \otimes \sigma'))(\nabla_B \otimes \nabla_{B'}) \\
&= d_{A \otimes A'}((\rho \otimes \rho') \otimes (\sigma \otimes \sigma')) b_{B, B', B, B'}(\nabla_B \otimes \nabla_{B'}) \quad ((D3), (M18)) \\
&= d_{A \otimes A'}((\rho \otimes \rho') \otimes (\sigma \otimes \sigma')) \nabla_{B \otimes B'} \quad ((\nabla 5)),
\end{aligned}$$

therefore $\rho \otimes \rho' \leq \sigma \otimes \sigma'$.

Now let λ and μ be morphisms from A into B . Then

$$\begin{aligned}
d_A(\lambda \otimes \mu) \nabla_B &= d_A(d_A(\lambda \otimes \lambda) \nabla_B \otimes \mu) \nabla_B \quad ((D\nabla)) \\
&= d_A(\lambda \otimes d_A(\lambda \otimes \mu) \nabla_B) \nabla_B \quad ((D1), (M6), (\nabla 3)) \\
&= d_{A s_{A,A}}(\lambda \otimes d_A(\lambda \otimes \mu) \nabla_B) \nabla_B \quad ((D2)) \\
&= d_A(d_A(\lambda \otimes \mu) \nabla_B \otimes \lambda) s_{B,B} \nabla_B \quad ((M8)) \\
&= d_A(d_A(\lambda \otimes \mu) \nabla_B \otimes \lambda) \nabla_B \quad ((\nabla 4)),
\end{aligned}$$

hence $d_A(\lambda \otimes \mu)\nabla_B \leq \lambda$. In the same manner one shows $d_A(\lambda \otimes \mu)\nabla_B \leq \mu$.

Further let be $\tau \leq \lambda$ and $\tau \leq \mu$. Then it follows

$$\tau = d_A(\tau \otimes \mu)\nabla_B = d_A(d_A(\tau \otimes \lambda)\nabla_B \otimes \mu)\nabla_B = d_A(\tau \otimes d_A(\lambda \otimes \mu)\nabla_B)\nabla_B,$$

therefore $\tau \leq d_A(\lambda \otimes \mu)\nabla_B$. Consequently, $d_A(\lambda \otimes \mu)\nabla_B$ is the greatest lower bound of λ and μ with respect to the partial order relation. ■

Lemma 2.3. *Any $hdht\nabla$ -category \underline{K} has the following properties:*

$$\begin{aligned} \forall A \in |K| & \quad (\nabla_A d_A \leq 1_{A \otimes A}), \\ \forall A, A' \in |K| \forall \rho \in K[A, A'] & \quad (\rho d_{A'} \leq d_A(\rho \otimes \rho)), \\ \forall A, A' \in |K| \forall \rho \in K[A, A'] & \quad (\nabla_A \rho \leq (\rho \otimes \rho)\nabla_{A'}). \end{aligned}$$

Proof. Composing the equation in condition (V2) with $\nabla_{A', A'}$ and using (V1) one obtains

$$\nabla_A d_A = \nabla_A d_A d_{A \otimes A} \nabla_{A' \otimes A'} = d_{A \otimes A} (\nabla_A d_A \otimes 1_{A \otimes A}) \nabla_{A \otimes A},$$

hence $\nabla_A d_A \leq 1_{A \otimes A}$ by the definition of \leq .

Condition (D ∇) gives rise to

$$\rho d_{A'} = (d_A(\rho \otimes \rho)\nabla_{A'})d_{A'} = (d_A(\rho \otimes \rho))(\nabla_{A'} d_{A'}) \leq d_A(\rho \otimes \rho) \quad \text{and}$$

$$\nabla_A \rho = \nabla_A (d_A(\rho \otimes \rho)\nabla_{A'}) = (\nabla_A d_A)((\rho \otimes \rho)\nabla_{A'}) \leq (\rho \otimes \rho)\nabla_{A'},$$

respectively. ■

Corollary 2.4. *By the definition of the partial order relation,*

$$(D9') \quad \rho d_{A'} = d_A(\rho d_{A'} \otimes d_A(\rho \otimes \rho))\nabla_{A' \otimes A'} \text{ and}$$

$$(V9') \quad \nabla_A \rho = d_{A \otimes A}(\nabla_A \rho \otimes (\rho \otimes \rho)\nabla_{A'})\nabla_{A'}$$

are identities in each $hdht\nabla$ -category \underline{K} . ■

Theorem 2.5. *Let \underline{K} be an $hdht\nabla$ -category as defined above. Then the class*

$$F^K := \{\rho \in K \mid d_{\text{dom } \rho}(\rho \otimes \rho) = \rho d_{\text{cod } \rho}\}$$

of so-called functional morphisms forms an $hdht\nabla$ -subcategory \underline{F}^K of \underline{K} which is even a $dht\nabla$ -category.

The partial order relation in the $dht\nabla$ -symmetric category \underline{F}^K is the restriction of \leq in the $hdht\nabla$ -symmetric category \underline{K} .

Proof. The conditions (D5), (D7), and (D8) show that the class F^K contains all morphisms of the families d , t , and ∇ , respectively.

Let $\rho \in K[A, B]$ be an isomorphism in \underline{K} . Then there is a $\rho^{-1} \in K[B, A]$ such that $\rho^{-1}d_A \leq d_B(\rho^{-1} \otimes \rho^{-1})$ and $\rho d_B \leq d_A(\rho \otimes \rho)$, hence $d_A(\rho \otimes \rho) \leq \rho d_B \leq d_A(\rho \otimes \rho)$, i.e. $\rho d_B = d_A(\rho \otimes \rho)$. Therefore, each isomorphism of \underline{K} belongs to F^K , especially, all identities and all morphisms of the families a , a^{-1} , r , r^{-1} , l , l^{-1} , s , s^{-1} , b , b^{-1} are in F^K . All zero morphisms $o_{A,B}$, $A, B \in |K|$, $o = o_{I,O}$, are elements of F^K since $o_{A,B}d_B = o_{A,B \otimes B} = d_A(o_{A,B} \otimes o_{A,B})$.

Let $\rho \in K[A, B] \cap F^K$ and $\sigma \in K[B, C] \cap F^K$. Then

$$(\rho\sigma)d_C = \rho(\sigma d_C) = \rho(d_B(\sigma \otimes \sigma)) = (\rho d_B)(\sigma \otimes \sigma) = d_A(\rho \otimes \rho)(\sigma \otimes \sigma) = d_A(\rho\sigma \otimes \rho\sigma),$$

hence F^K is closed under composition.

If $\rho \in K[A, B]$ and $\rho' \in K[A', B']$ are morphisms of F^K , then $(\rho \otimes \rho') \in K[A \otimes A', B \otimes B']$ is in F^K too, since

$$\begin{aligned} (\rho \otimes \rho')d_{B \otimes B'} &= (\rho \otimes \rho')(d_B \otimes d_{B'})b_{B,B',B'} \\ &= (d_A(\rho \otimes \rho) \otimes d_{A'}(\rho' \otimes \rho'))b_{B,B',B'} \\ &= (d_A \otimes d_{A'})b_{A,A',A'}((\rho \otimes \rho') \otimes (\rho \otimes \rho')) \\ &= d_{A \otimes A'}((\rho \otimes \rho') \otimes (\rho \otimes \rho')). \end{aligned}$$

With respect to the axioms of an $hdht\nabla s$ -category, which are identities only, and because of the defining condition of $F^K \subseteq K$, one has a $dht\nabla s$ -category \underline{F}^K .

The partial order relation \leq in \underline{K} is defined by $\rho \leq \sigma \Leftrightarrow \rho = d_A(\rho \otimes \sigma)\nabla_{A'}$ for morphisms $\rho, \sigma \in K[A, A']$. By property (P ∇), this condition is equivalent to $\rho = d_A(\rho \otimes \sigma)p_2^{A',A'}$ for morphisms ρ, σ of F^K , hence $\rho \leq \sigma$ with respect to the partial order relation in the $dht\nabla s$ -category \underline{F}^K . ■

Proposition 2.6. *All morphisms $\rho \in K[A, B]$, $A, B \in |K|$, of an $hdht\nabla s$ -category \underline{K} fulfilling the condition $\rho t_B = t_A$ (so-called total morphisms) form a symmetric monoidal subcategory $T^{K\bullet}$ which contains all retractions of \underline{K} and all morphisms t_A , $A \in |K|$.*

Moreover, $\underline{T}^K := (T^{K\bullet}, d, t)$ is an $hdts$ -category.

Proof. Obviously, all identity morphisms 1_A , $A \in |K|$, are in T^K . Because of

$$\rho t_B = t_A \wedge \sigma t_C = t_B \Rightarrow (\rho\sigma)t_C = \rho(\sigma t_C) = \rho t_B = t_A$$

and

$$\rho t_B = t_A \wedge \rho' t_{B'} = t_{A'} \Rightarrow (\rho \otimes \rho') t_{B \otimes B'} = (\rho \otimes \rho')(t_B \otimes t_{B'}) t_{I \otimes I} = (t_A \otimes t_{A'}) t_{I \otimes I} = t_{A \otimes A'}$$

the class T^K is closed under composition and \otimes -operation.

Let $\rho \in K[A, B]$ be a coretraction in \underline{K} . Then there is $\rho^* \in K[B, A]$ such that $\rho \rho^* = 1_A$. So, one has (see [6], p. 12)

$$\rho t_B = 1_A \rho t_B = d_A(1_A \otimes t_A) r_A \rho t_B \quad ((T1))$$

$$= d_A(\rho t_B \otimes t_A) r_I \quad ((M7))$$

$$= d_A(\rho \otimes \rho)(t_B \otimes \rho^* t_A) r_I \quad ((\rho \rho^* = 1_A))$$

$$\geq \rho d_B(t_B \otimes 1_B)(1_I \otimes \rho^* t_A) l_I \quad ((2.3))$$

$$= \rho d_b(t_B \otimes 1_B) l_B \rho^* t_A \quad ((M14))$$

$$= \rho 1_B \rho^* t_A \quad ((T4))$$

$$= t_A \geq \rho t_B,$$

therefore $\rho t_B = t_A$, hence $\rho \in T^K$.

Because of $t_A t_I = t_A 1_I = t_A$, $A \in |K|$, $d_A \nabla_A = 1_A$, $A \in |K|$, and each isomorphism is just a coretraction, all morphisms of the families a , a^{-1} , r , r^{-1} , l , l^{-1} , s , s^{-1} , b , b^{-1} , d , and t belong to T^K .

Since arbitrary suitable morphisms and objects of \underline{K} fulfil the identities (D1), (D2), (D3), (D5), (D6), (D7), (T1), (T2), (T3), (T4), (T5), (T6), (T7), (T8), (T9), the sequence (T^{K^\bullet}, d, t) is an *hdts*-category. ■

Corollary 2.7. *Let \underline{K} be any $hdht\nabla s$ -category. Then all morphisms of the families 1 , a , r , l , s , b , d , t , ∇ , and $(o_{A,B} \mid A, B \in |K|)$ possess all properties of such morphisms in a $dht\nabla s$ -category, especially the following identities are valid:*

$$(D8), (T4), (T5), (T7), (T8), (B1), (B2), (o3), (o4), (o5),$$

$$(\nabla6), (\nabla7), (\nabla8), (\nabla10),$$

$$(I1) \quad \nabla_I d_I = 1_{I \otimes I},$$

$$(I2) \quad t_{I \otimes I} = \nabla_I = l_I = r_I = d_I^{-1},$$

$$(I3) \quad d_I = r_I^{-1} = l_I^{-1},$$

$$(I4) \quad d_I \otimes d_I = d_{I \otimes I}. \quad \blacksquare$$

Lemma 2.8. *Let \underline{K} be an $hdht\nabla s$ -category. Then one has*

$$(T9) \quad \rho t_{A'} d_I = d_A(\rho t_{A'} \otimes t_A)$$

for all objects $A, A' \in |K|$ and all morphisms $\rho \in K[A, A']$.

Moreover:

- (i) $\forall A, A' \in |K| \forall \rho \in K[A, A']$ ($\rho d_{A'} d_{A' \otimes A'} = d_A(\rho d_{A'} \otimes d_A(\rho \otimes \rho))$
 $\Rightarrow \rho d_{A'} = d_A(\rho d_{A'} \otimes d_A(\rho \otimes \rho)) \nabla_{A' \otimes A'}$),
- (ii) $\forall A, A' \in |K| \forall \rho \in K[A, A']$ ($\nabla_A \rho d_{A'} = d_A(\nabla_A \rho \otimes (\rho \otimes \rho) \nabla_{A'})$
 $\Rightarrow \nabla_A \rho = d_A(\nabla_A \rho \otimes (\rho \otimes \rho) \nabla_{A'}) \nabla_{A'}$),
- (iii) $\forall A, A' \in |K| \forall \rho \in K[A, A']$ ($\rho t_{A'} d_I = d_A(\rho t_{A'} \otimes t_A)$
 $\Leftrightarrow \rho t_{A'} = d_A(\rho t_{A'} \otimes t_A) \nabla_I$).

Proof. Because of $\nabla_I d_I = 1_{I \otimes I}$ and $\nabla_I = r_I = l_I = t_{I \otimes I}$ the equation

$$\begin{aligned} d_A(\rho t_{A'} \otimes t_A) &= d_A(\rho t_{A'} \otimes t_A) \nabla_I d_I = d_A(\rho t_{A'} \otimes t_A) r_I d_I \\ &= d_A(1_A \otimes t_A) r_A \rho t_{A'} d_I = \rho t_{A'} d_I \end{aligned}$$

is valid for each $\rho \in K[A, A']$ and all $A, A' \in |K|$, hence \underline{K} fulfils condition (T9).

The condition (T9') is equivalent to (T9), since

$$d_A(\rho t_{A'} \otimes t_A) = \rho t_{A'} d_I \Rightarrow d_A(\rho t_{A'} \otimes t_A) \nabla_I = \rho t_{A'}$$

by $d_I \nabla_I = 1_I$ and

$$d_A(\rho t_{A'} \otimes t_A) \nabla_I = \rho t_{A'} \Rightarrow d_A(\rho t_{A'} \otimes t_A) = \rho t_{A'} d_I$$

by $\nabla_I d_I = 1_{I \otimes I}$, hence property (iii) is shown.

The implications (i) and (ii) are satisfied because of the general property

$$\xi d_B = d_A(\xi \otimes \eta) \Rightarrow \xi = \xi d_B \nabla_B = d_A(\xi \otimes \eta) \nabla_B.$$

■

Remark 2.9. The opposite of the implications (i) and (ii), respectively, is not true in general, since there are counterexamples in Rel.

Remark 2.10. As in any $dht\nabla s$ -category, the morphisms

$$\begin{aligned} p_1^{A,B} &:= (1_A \otimes t_B)r_A \in K[A \otimes B, A] \cap F^K, \\ p_2^{A,B} &:= (t_A \otimes 1_B)l_B \in K[A \otimes B, B] \cap F^K \end{aligned}$$

of an arbitrary $hdht\nabla s$ -category \underline{K} are called *canonical projections* again and one has

$$\nabla_A = \inf \{p_1^{A,A}, p_2^{A,A}\} = d_A(p_1^{A,A} \otimes p_2^{A,A}) \nabla_A$$

for all $A \in |K|$.

Remark that $(A \otimes B; p_1^{A,B}, p_2^{A,B})$ is not a categorical product in the whole category \underline{K} , but in the subcategory T^K

The family $\nabla = (\nabla_A \mid A \in |K|)$ is uniquely determined by the family $d = (d_A \mid A \in |K|)$ and the conditions $(\nabla 1)$ and $(\nabla 2)$.

Lemma 2.11. *Let \underline{K} be an arbitrary $hdht\nabla s$ -category. Then there holds:*

- (*2) $\forall A, B, C \in |K| \forall \rho, \rho' \in K[A, B] \forall \sigma, \sigma' \in K[B, C] (d_A(\rho \otimes \rho')\nabla_B = \rho \wedge d_B(\sigma \otimes \sigma')\nabla_C = \sigma \Rightarrow d_A(\rho\sigma \otimes \rho'\sigma')\nabla_C = \rho\sigma)$,
- (*3) $\forall A, B \in |K| \forall \rho, \sigma \in K[A, B] (d_A(\rho \otimes \sigma)\nabla_B = \rho \wedge d_A(\sigma \otimes \sigma) = \sigma d_B \Rightarrow d_A(\rho \otimes \sigma)p_i^{B,B} = \rho \ (i \in \{1, 2\}))$,
- (*4) $\forall A, B \in |K| \forall \rho \in K[A, B] (d_A(\rho \otimes \rho)p_i^{B,B} = \rho \ (i \in \{1, 2\}))$,
- (*5) $\forall A, B \in |K| \forall \rho, \sigma \in K[A, B] (d_A(\rho \otimes \sigma)\nabla_B = \rho \wedge d_A(\sigma \otimes \sigma) = \sigma d_B \Rightarrow d_A(\rho \otimes \rho) = \rho d_B)$,
- (*6) $\forall A \in |K| \forall \rho \in K[A, A] (d_A(1_A \otimes \rho)\nabla_A = \rho \Rightarrow d_A(1_A \otimes \rho)p_1^{A,A} = d_A(1_A \otimes \rho)p_2^{A,A} = \rho)$.

Proof. Axiom $(*1)$ implies condition $(*2)$ because of $\rho \leq \rho' \wedge \sigma \leq \sigma' \Rightarrow \rho\sigma \leq \rho'\sigma'$. To show $(*3)$ note that $d_A(\rho \otimes \sigma)\nabla_B = \rho \Leftrightarrow \rho \leq \sigma$ and $d_A(\sigma \otimes \sigma) = \sigma d_B \Leftrightarrow \sigma \in F^K$. So one obtains

$$\begin{aligned}
d_A(\rho \otimes \sigma)p_i^{B,B} &= d_A(d_A(\rho \otimes \sigma)\nabla_B \otimes \sigma)p_i^{B,B} && (\rho \leq \sigma) \\
&= d_A(\rho \otimes d_A(\sigma \otimes \sigma))a_{B,B,B}(\nabla_B \otimes 1_B)p_i^{B,B} && (\sigma \in F_K) \\
&= d_A(\rho \otimes \sigma)(1_B \otimes d_B)a_{B,B,B}(\nabla_B \otimes 1_B)p_i^{B,B} && ((F4)) \\
&= d_A(\rho \otimes \sigma)\nabla_B d_B p_i^{B,B} && ((\nabla 7)) \\
&= d_A(\rho \otimes \sigma)\nabla_B = \rho
\end{aligned}$$

with respect to the axioms of an $hdht\nabla s$ -category.

The property (*4) is a consequence of (D9') and (T9'):

$$\begin{aligned}
\rho &= \rho d_B p_i^{B,B} \leq d_A(\rho \otimes \rho)p_i^{B,B} \wedge \rho t_B \leq t_A \\
\Rightarrow d_A(\rho \otimes \rho)p_1^{B,B} &= d_A(\rho \otimes \rho t_B)r_B \leq d_A(\rho \otimes t_A)r_B = d_A(1_A \otimes t_A)r_A \rho = \rho \\
\wedge d_A(\rho \otimes \rho)p_2^{B,B} &= d_A(\rho t_B \otimes \rho)l_B \leq d_A(t_A \otimes \rho)l_B = d_A(t_A \otimes 1_A)l_A \rho = \rho.
\end{aligned}$$

(*5): Using the previous results and the assumption one obtains

$$\begin{aligned}
d_A(\rho \otimes \rho) &= d_A(d_A(\rho \otimes \sigma)p_2^{B,B} \otimes d_A(\rho \otimes \sigma))p_2^{B,B} \\
&= d_A(d_A \otimes d_A)((\rho \otimes \sigma) \otimes (\rho \otimes \sigma))(p_2^{B,B} \otimes p_2^{B,B}) \\
&= d_A d_{A \otimes A}((\rho \otimes \sigma) \otimes (\rho \otimes \sigma))(p_2^{B,B} \otimes p_2^{B,B}) \\
&= d_A(d_A(\rho \otimes \rho) \otimes d_A(\sigma \otimes \sigma))b_{B,B,B,B}(p_2^{B,B} \otimes p_2^{B,B}) \\
&= d_A(d_A(\rho \otimes \rho) \otimes \sigma d_B)p_2^{B \otimes B, B \otimes B} \\
&= d_A(\rho \otimes d_A(\rho \otimes \sigma))a_{B,B,B}(1_{B \otimes B} \otimes d_B)p_2^{B \otimes B, B \otimes B} \\
&= d_A(\rho \otimes d_A(\rho \otimes \sigma))a_{B,B,B}p_2^{B \otimes B, B} d_B \\
&= d_A(\rho \otimes d_A(\rho \otimes \sigma))(1_B \otimes p_2^{B,B})p_2^{B,B} d_B \\
&= d_A(\rho \otimes d_A(\rho \otimes \sigma))p_2^{B,B} p_2^{B,B} d_B \\
&= d_A(\rho \otimes \rho)p_2^{B,B} d_B = \rho d_B.
\end{aligned}$$

The property (*6) arises from (*3) because of $1_A \in F^K$ for each $A \in |K|$. ■

Lemma 2.12. *Let \underline{K} be a monoidal symmetric category endowed with morphisms families $d, t, (o_{A,B} \mid A, B \in |K|)$, and ∇ such that all axioms of an $hdht\nabla s$ -category without (*1) are fulfilled. Moreover, let the condition (*2) be valid. Then \underline{K} is an $hdht\nabla s$ -category in the defined sense as above.*

Proof. It remains to show the condition (*1):

$$\begin{aligned}
& d_A(d_A(\rho \otimes \rho')\nabla_B d_B(\sigma \otimes \sigma')\nabla_C \otimes d_A(\rho\sigma \otimes \rho'\sigma')\nabla_C)\nabla_C \\
&= d_A(\rho\sigma \otimes d_A(\rho\sigma \otimes \rho'\sigma')\nabla_C)\nabla_C && ((*2)) \\
&= d_A(1_A \otimes d_A)(\rho\sigma \otimes (\rho\sigma \otimes \rho'\sigma'))(1_C \otimes \nabla_C)\nabla_C && ((F4)) \\
&= d_A(d_A \otimes 1_A)a_{A,A,A}^{-1}(\rho\sigma \otimes (\rho\sigma \otimes \rho'\sigma'))(1_C \otimes \nabla_C)\nabla_C && ((D3)) \\
&= d_A(d_A \otimes 1_A)((\rho\sigma \otimes \rho\sigma) \otimes \rho'\sigma')a_{C,C,C}^{-1}(1_C \otimes \nabla_C)\nabla_C && ((M6)) \\
&= d_A(d_A)(\rho\sigma \otimes \rho\sigma) \otimes \rho'\sigma')(\nabla_C \otimes 1_C)\nabla_C && ((\nabla3)) \\
&= d_A(d_A(\rho\sigma \otimes \rho\sigma)\nabla_C \otimes \rho'\sigma')\nabla_C && ((F4)) \\
&= d_A(\rho\sigma \otimes \rho'\sigma')\nabla_C && ((D\nabla)) \\
&= \rho\sigma && ((*2)) \\
&= d_A(\rho \otimes \rho')\nabla_B d_B(\sigma \otimes \sigma')\nabla_C && ((*2))
\end{aligned}$$

■

The results of the last both lemmata are important for the axiomization of $hdht\nabla s$ -categories. The system of axioms for an $hdht\nabla s$ -category given in [11] contains two identical implications, namely (21) (\Leftrightarrow (*2)) and (20) (\Leftrightarrow (*6)). The property (*6) is a consequence of the other properties and the conditions (*1) and (*2) are equivalent in a monoidal symmetric category \underline{K} endowed with morphisms families $d, t, (o_{A,B} \mid A, B \in |K|)$, and ∇ such that

$$\begin{aligned}
& (D1), (D2), (D3), (D5), (D7), (D8), (T1), (T2), (T6), (T9'), \\
& (\nabla1), (\nabla2), (\nabla3), (\nabla4), (\nabla5), (\nabla6), (\nabla7), (D\nabla), \\
& (o1), (o2), (O1)
\end{aligned}$$

are fulfilled. Therefore, $hdht\nabla s$ -categories are axiomatizable by identities only, hence all small $hdht\nabla s$ -categories form a variety of many-sorted total

algebras and there are free many-sorted algebras to each generating set with respect to this variety. Especially, there are free $hdht\nabla s$ -theories, i.e. free algebraic theories for relational structures, by analogy with the existence of free algebraic theories for partial algebras ([3], [10]).

Lemma 2.13. *In any $hdht\nabla$ -symmetric category the following conditions are fulfilled for arbitrary morphisms ρ, σ :*

$$(j) \quad \rho\sigma = 1_A \quad \wedge \quad \sigma\rho \leq 1_B \quad \Rightarrow \quad d_A(\rho \otimes \rho) = \rho d_B$$

$$(jj) \quad \rho\sigma \leq 1_A \quad \wedge \quad \sigma\rho = 1_B \quad \Rightarrow \quad \nabla_A \rho = (\rho \otimes \rho) \nabla_B$$

Proof. To show (j) we use at first the known property $\sigma d_A \leq d_B(\sigma \otimes \sigma)$. Further,

$$d_A(\rho \otimes \rho) = \rho \sigma d_A(\rho \otimes \rho) \leq \rho d_B(\sigma \otimes \sigma)(\rho \otimes \rho) \leq \rho d_B(1_B \otimes 1_B) = \rho d_B,$$

hence $d_A(\rho \otimes \rho) = \rho d_B$ by $\rho d_B \leq d_A(\rho \otimes \rho)$.

In a similar way one shows the statement (jj), namely because of $\nabla_B \sigma \leq (\sigma \otimes \sigma) \nabla_A$ and

$$(\rho \otimes \rho) \nabla_B = (\rho \otimes \rho) \nabla_B \sigma \rho \leq (\rho \sigma \otimes \rho \sigma) \nabla_A \rho \leq \nabla_A \rho \leq (\rho \otimes \rho) \nabla_B$$

one has $\nabla_A \rho = (\rho \otimes \rho) \nabla_B$. ■

Definition 2.14. Morphisms $e \in K[A, A] \subseteq K$ with the property $e \leq 1_A$, i.e. $e = d_A(1_A \otimes e) \nabla_A$, are called *subidentities* in \underline{K} (compare with ([7])).

Proposition 2.15 (cf. [7]). *For each morphism $\rho : A \rightarrow B$, $A, B \in |K|$, the morphism*

$$\alpha(\rho) := d_A(\rho \otimes 1_A) p_2^{B,A}$$

is a subidentity of A in \underline{K} and there holds $\alpha(\rho)\rho = \rho$. Each subidentity e of \underline{K} fulfils $d_A(e \otimes e) = ed_A$, therefore the subidentities of \underline{K} are the subidentities of \underline{F}^K and satisfy the following conditions for all suitable morphisms and objects of K :

- (e1) $e \leq 1_A \Rightarrow ee = e,$
(e2) $e_1, e_2 \leq 1_A \Rightarrow e_1 e_2 = e_2 e_1 = \inf\{e_1, e_2\},$
(e3) $e_1 \leq e_2 \leq 1_A \Leftrightarrow e_1 = e_1 e_2 \leq 1_A,$
(e4) $e \leq 1_A \Leftrightarrow \alpha(e) = e,$
(e5) $e \leq 1_A \Rightarrow ed_A = d_A(e \otimes e) = d_A(e \otimes 1_A),$
(e6) $e \leq 1_A \Rightarrow \nabla_A e = (e \otimes e) \nabla_A = (e \otimes 1_A) \nabla_A,$
(e7) $\rho, \sigma \in K[A, B] \Rightarrow \alpha(\rho)\sigma = d_A(\rho \otimes \sigma) p_2^{B,B} \wedge \alpha(\sigma)\rho = d_A(\rho \otimes \sigma) p_1^{B,B},$
(e8) $\alpha(\rho)\sigma = \rho \Rightarrow \rho \leq \sigma,$
(e9) $e\rho = \rho \wedge e \leq 1_A \Leftrightarrow \alpha(\rho) \leq e \leq 1_A,$
(e10) $\text{cod}\rho = \text{dom}\sigma \Rightarrow \alpha(\rho\sigma) \leq \alpha(\rho),$
(e11) $e \leq 1_A \Rightarrow \alpha(e\rho) \leq e,$
(e12) $e \leq 1_A \Rightarrow \alpha(e\rho) = e\alpha(\rho),$
(e13) $\rho \leq \sigma \Rightarrow \alpha(\rho) \leq \alpha(\sigma),$
(e14) $\text{cod}\rho = \text{dom}\sigma \Rightarrow \rho\alpha(\sigma) \leq \alpha(\rho\sigma)\rho,$
(e15) $\text{cod}\rho = \text{dom}\sigma \Rightarrow \alpha(\rho\sigma) = \alpha(\rho\alpha(\sigma)),$

Proof. Because of $\rho t_B \leq t_A$ one obtains

$$\alpha(\rho) = d_A(\rho \otimes 1_A) p_2^{B,A} = d_A(\rho t_B \otimes 1_A) l_A \leq d_A(t_A \otimes 1_A) l_A = 1_A.$$

Using the definition of $\alpha(\rho)$, properties (M14), (M15), and $\alpha(\rho) \leq 1_A$ one receives $\alpha(\rho)\rho = \rho$ via

$$\alpha(\rho)\rho = d_A(\rho \otimes 1_A) p_2^{B,A} \rho = d_A(\rho \otimes \rho) p_2^{B,B} \geq \rho d_{BP_2}^{B,B} = \rho = 1_A \rho \geq \alpha(\rho)\rho.$$

Because of $e \leq 1_A$ the property $d_A(e \otimes e) = ed_A$ is a consequence of Lemma 2.11, (*5), and the subidentities of \underline{K} are exactly the subidentities of \underline{F}^K , therefore, all subidentities have the properties (e1), (e2), (e3) and (e4) (cf. [7]).

To show property (e5) use the property (e4) $e \leq 1_A \Rightarrow e = \alpha(e) = d_A(e \otimes 1_A)p_2^{A,A}$:

$$\begin{aligned} d_A(e \otimes e) &= d_A(e \otimes d_A(e \otimes 1_A)p_2^{A,A}) = d_A(d_A(e \otimes e) \otimes 1_A)a_{A,A,A}^{-1}(1_A \otimes p_2^{A,A}) \\ &= d_A(d_A(e \otimes e)p_1^{A,A} \otimes 1_A) = d_A(e \otimes 1_A). \end{aligned}$$

The second part of the property (e6) is a consequence of (e2) and (e5) owing to $\nabla_A d_A \leq 1_{A \otimes A}$, $(e \otimes e) \leq 1_{A \otimes A}$, and $(e \otimes 1_A) \leq 1_{A \otimes A}$:

$$\begin{aligned} d_A(e \otimes e) = d_A(e \otimes 1_A) &\Rightarrow \nabla_A d_A(e \otimes e) = \nabla_A d_A(e \otimes 1_A) \\ &\Rightarrow (e \otimes e)\nabla_A d_A = (e \otimes 1_A)\nabla_A d_A \quad ((e2)) \\ &\Rightarrow (e \otimes e)\nabla_A d_A \nabla_A = (e \otimes 1_A)\nabla_A d_A \nabla_A \\ &\Rightarrow (e \otimes e)\nabla_A = (e \otimes 1_A)\nabla_A. \quad ((\nabla 1)) \end{aligned}$$

Because of $(e \otimes e) \leq 1_{A \otimes A}$ and $\nabla_A d_A \leq 1_{A \otimes A}$ one has

$$\begin{aligned} (e \otimes e)\nabla_A &= (e \otimes e)\nabla_A d_A \nabla_A \quad (d_A \nabla_A = 1_A) \\ &= \nabla_A d_A (e \otimes e) \nabla_A \quad ((e2)) \\ &= \nabla_A e. \quad ((D\nabla)) \end{aligned}$$

Property (e7) is an immediate consequence of (M7), (M14), (M8), and (M13).

To show (e8) take into consideration

$$\rho = \alpha(\rho)\sigma \leq 1_A \sigma = \sigma.$$

(e9): Assuming $e\rho = \rho$, $e \leq 1_A$ one gets

$$\alpha(\rho) = \alpha(e\rho) = d_A(e\rho \otimes 1_A)p_2^{B,A} = d_A(e\rho t_B \otimes 1_A)l_A \leq d_A(e t_A \otimes 1_A)l_A = \alpha(e) = e.$$

Conversely, $\alpha(\rho) \leq e \leq 1_A$ yields

$$\rho = \alpha(\rho)\rho \leq e\rho \leq 1_A \rho = \rho.$$

Condition (e10) is true, since

$$\alpha(\rho\sigma) = d_A(\rho\sigma \otimes 1_A)p_2^{C,A} = d_A(\rho\sigma t_C \otimes 1_A)l_A \leq d_A(\rho t_B \otimes 1_A)l_A = \alpha(\rho).$$

Condition (e11) arises from $\alpha(e\rho) \leq \alpha(e) = e$.

Property (e12) is a consequence of (e5) as follows:

$$\begin{aligned}\alpha(e\rho) &= d_A(e\rho \otimes 1_A)p_2^{B,A} = d_A(e \otimes 1_A)(\rho \otimes 1_A)p_2^{B,A} \\ &= d_A(e \otimes e)(\rho \otimes 1_A)p_2^{B,A} = ed_A(\rho \otimes 1_A)p_2^{B,A} \\ &= e\alpha(\rho).\end{aligned}$$

To show (e13) use the definitions of \leq and $\alpha(\rho)$ ($\rho : A \rightarrow B$, $\sigma : B \rightarrow C$):

$$\begin{aligned}\alpha(\rho) &= d_A(\rho \otimes 1_A)p_2^{B,A} = d_A(d_A(\rho \otimes \sigma)\nabla_B \otimes 1_A)p_2^{B,A} && (\rho \leq \sigma) \\ &\leq d_A(d_A(\rho \otimes \sigma)p_2^{B,B} \otimes 1_A)p_2^{B,A} && (\nabla_B \leq p_2^{B,B}) \\ &= d_A(d_A(\rho \otimes 1_A)p_2^{B,A}\sigma \otimes 1_A)p_2^{B,A} && ((M14)) \\ &= d_A(\alpha(\rho)\sigma \otimes 1_A)p_2^{B,A} \\ &\leq d_A(\sigma \otimes 1_A)p_2^{B,A} = \alpha(\sigma). && (\alpha(\rho)\sigma \leq \sigma)\end{aligned}$$

Assertion (e14) is true since

$$\rho\alpha(\sigma) = \rho d_B(\sigma \otimes 1_B)p_2^{C,B} \leq d_A(\rho\sigma \otimes \rho)p_2^{C,B} = \alpha(\rho\sigma)\rho.$$

Condition (e15) follows by (e10), (e13), and (e14):

Let ρ and σ be as above. Then one has

$$\alpha(\rho\sigma) = \alpha(\rho\alpha(\sigma)\sigma) \leq \alpha(\rho\alpha(\sigma)),$$

hence

$$\begin{aligned}\alpha(\rho\sigma) &\leq \alpha(\rho\alpha(\sigma)) \leq \alpha(\alpha(\rho\sigma)\rho) \leq \alpha(\alpha(\rho\sigma)\alpha(\rho)) \\ &\leq \alpha(\alpha(\rho\sigma)1_A) = \alpha(\alpha(\rho\sigma)) = \alpha(\rho\sigma).\end{aligned}$$

■

Remark that, as an easy example shows, in *Rel* the opposite implication to (e8) is not true: Let be given $A = \{a\}$, $B = \{b_1, b_2\}$, $\rho = \{(a, b_1)\}$, $\sigma = \{(a, b_1), (a, b_2)\}$. Then $\rho \leq \sigma$ and $\rho < \alpha(\rho)\sigma = \sigma$.

Furthermore, the equality in (e14) is not true in general. For this let be the sets A and B as above and let be $C = \{x\}$. For the relations σ as above and $\tau = \{(b_1, x)\}$ one obtains $\sigma\alpha(\tau) = \{(a, b_1)\}$ and $\sigma\tau = \{(a, x)\}$, hence $\alpha(\sigma\tau) = \{(a, a)\}$, consequently $\alpha(\sigma\tau)\sigma = \{(a, b_1), (a, b_2)\} = \sigma \neq \sigma\alpha(\tau)$.

REFERENCES

- [1] L. Budach and H.-J. Hoehnke, *Automaten und Funktoren*, Akademie-Verlag, Berlin 1975.
- [2] S. Eilenberg and G.M. Kelly, *Closed categories*, p. 421–562 in: “*Proceedings of the Conference on Categorical Algebra (La Jolla, 1965)*”, Springer-Verlag, New York 1966.
- [3] H.-J. Hoehnke, *On partial algebras*, p. 373–412 in: *Colloquia Mathematica Societatis János Bolyai*, 29 (“*Universal Algebra, Esztergom (Hungary) 1977*”), North-Holland, Amsterdam 1981.
- [4] G.M. Kelly, *On MacLane’s condition for coherence of natural associativities, commutativities, etc.*, *J. Algebra* **1** (1964), 397–402.
- [5] S. MacLane, *Kategorien, Begriffssprache und mathematische Theorie*, Springer-Verlag, Berlin-New York 1972.
- [6] J. Schreckenberger, *Über dht-symmetrische Kategorien*, Semesterarbeit Päd. Hochschule Köthen, Köthen 1978.
- [7] J. Schreckenberger, *Über die Einbettung von dht-symmetrischen Kategorien in die Kategorie der partiellen Abbildungen zwischen Mengen*, AdW DDR Berlin, Zentralinstitut f. Mathematik und Mechanik, Preprint P-12/8 (1980).
- [8] J. Schreckenberger, *Zur Theorie der dht-symmetrischen Kategorien*, Dissertation (B), eingereicht an der Math.-Naturwiss. Fak. d. Wiss. Rates d. Pädag. Hochschule ”Karl Liebknecht”, Potsdam 1984.
- [9] J. Schreckenberger, *Zur Axiomatik von Kategorien partieller Morphismen*, Beiträge Algebra Geom. (Halle/Saale) **24** (1987), 83–98.
- [10] H.-J. Vogel, *Eine kategorientheoretische Sprache zur Beschreibung von Birkhoff-Algebren*, AdW DDR, Institut f. Mathematik, Report R-MATH-06/84, Berlin (1984).
- [11] H.-J. Vogel, *Relations as morphisms of a certain monoidal category*, p. 205–217, in: “*General Algebra and Applications in Discrete Mathematics*”, Shaker Verlag, Aachen 1997.
- [12] H.-J. Vogel, *On functors between dht ∇ -symmetric categories*, *Discuss. Math. - Algebra & Stochastic Methods* **18** (1998), 131–147.
- [13] H.-J. Vogel, *On Properties of dht ∇ -symmetric categories*, *Contributions to General Algebra* **11** (1999), 211–223.
- [14] H.-J. Vogel, *Halfdiagonal-halfterminal-symmetric monoidal categories with inversions*, p. 189–204 in: “*General Algebra and Discrete Mathematics*”, Shaker Verlag, Aachen 1999.

- [15] H.-J. Vogel, *On morphisms between partial algebras*, p. 427–453 in: “*Algebras and Combinatorics. An International Congress, ICAC '97, Hong Kong*”, Springer-Verlag, Singapore 1999.

Received 6 December 2000