

ON THE SPECIAL CONTEXT OF INDEPENDENT SETS*

VLADIMÍR SLEZÁK

*Department of Algebra and Geometry,
Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic*
e-mail: slezak@prfnw.upol.cz

Abstract

In this paper the context of independent sets \mathcal{J}_L^p is assigned to the complete lattice $(\mathcal{P}(M), \subseteq)$ of all subsets of a non-empty set M . Some properties of this context, especially the irreducibility and the span, are investigated.

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Let us denote by (L, \leq) a complete lattice in which \vee, \wedge mean the supremum and the infimum of any subset of L , respectively. The least and the greatest elements in (L, \leq) are denoted by 0, 1, respectively. If $a, b \in L$, then $a \parallel b$ means that a, b are incomparable in (L, \leq) .

For a subset $A \subseteq L$ we put $U(A) = \{x \in L \mid (\forall a \in A)[a \leq x]\}$ and $L(A) = \{x \in L \mid (\forall a \in A)[x \leq a]\}$. Obviously, $U(A) = U(\vee A)$ and $L(A) = L(\wedge A)$. Moreover, let us put $|A| := \text{card } A$ and $A_a := A \setminus \{a\}$, $s_a := \vee A_a, i_a := \wedge A_a$ for all $a \in A$.

Definition 1 (F. Machala, [6]). A subset $A \subseteq L$ is said to be *join-independent* if and only if $a \not\leq s_a$ for all $a \in A$. A subset $B \subseteq L$ is said to be *meet-independent* if and only if $i_b \not\leq b$ for all $b \in B$.

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Remark 1. The concepts of join-independent and meet-independent sets are the special cases of the definition of independent sets in a context [†] (an incidence structure) or, more precisely, in two closure spaces associated to each context. Any complete lattice (L, \leq) (and even any partially ordered set) can be understood as the context (L, L, \leq) , where (under the denotation established for contexts) $A^\uparrow = U(A)$, $A^{\uparrow\downarrow} = LU(A)$ and $B^\downarrow = L(B)$, $B^{\downarrow\uparrow} = UL(B)$ for $A, B \subseteq L$. The closure operators are given by $A \mapsto A^{\uparrow\downarrow}$, $B \mapsto B^{\downarrow\uparrow}$ for $A, B \subseteq L$.

The notion of an independent set in a lattice appears in various approaches in literature (see [1], [3], [4], [7], [9] and [10]). In fact, in this paper irredundant sets in complete lattices are discussed, but we prefer to use the terms "join-independent" and "meet-independent" with respect to connections with closure systems and incidence structures.

Remark 2. The notions of join- and meet-independent sets are dual in complete lattices. In the following we will only investigate join-independent sets. Analogous results for meet-independent sets can be obtained dually.

Propositions 1–7 are easy consequences of the definitions of join- and meet-independencies. Thus, the proofs of them are omitted.

Proposition 1. *Every singleton $A = \{a\}$, $a \neq 0$, $a \in L$, is join-independent.*

Proposition 2. *A subset $A \subseteq L$, $|A| \geq 2$, is join-independent if and only if $a \parallel s_a$ for all $a \in A$.*

Proposition 3. *If a subset $A \subseteq L$ is join-independent, then $a \parallel b$ for all $a, b \in A$, $a \neq b$.*

Let us introduce one more denotation: If $A \subseteq L$, then for $a \in A$ we put $X^A(a) := U(s_a) \setminus U(a)$, $Y^A(a) := L(i_a) \setminus L(a)$.

Proposition 4. *If $A \subseteq L$ is join-independent, then $X^A(a) \cap X^A(b) = \emptyset$ for any $a, b \in A$, $a \neq b$.*

Proposition 5. *The subset $A \subseteq L$ is join-independent if and only if $X^A(a) \neq \emptyset$ for all $a \in A$.*

[†]A context is the triple (G, H, I) , where G and H are sets and $I \subseteq G \times H$ (see [2]).

Proposition 6. *If the subset $A \subseteq L$ is join-independent, then every choice $Q^A = \{m_a \in X^A(a) \mid a \in A\}$ is a meet-independent set.*

Remark 3. Let $A \subseteq L$ be join-independent. Then for any choice $Q^A = \{m_a \in X^A(a) \mid a \in A\}$ the mapping $\alpha : a \mapsto m_a$ is a one-to-one mapping of the join-independent set A onto the meet-independent set Q^A . Analogously for a meet-independent subset. This is called a *norming mapping* of the set A (see [5]). If we denote by L_j^p (L_m^p) the set of all p -element join-independent (meet-independent) sets of (L, \leq) (p is any cardinal number), then it is possible to define the *context of independent sets* $\mathcal{J}_L^p = (L_j^p, L_m^p, I^p)$, where the relation I^p is given by the following: For $A \in L_j^p, B \in L_m^p$ we put AI^pB if and only if there exists a norming mapping $\alpha : A \rightarrow B$. (If $L_j^p = \emptyset$, then $L_m^p = \emptyset$ and $\mathcal{J}_L^p = (\emptyset, \emptyset, \emptyset)$.) If $A \in L_j^p$, then obviously AI^pS_A where $S_A = \{s_a \mid a \in A\}$.

Proposition 7. *If a set $A \subseteq L$ is join-independent, then every subset of A is join-independent.*

Now we recall some basic notions from the general theory of contexts (see [8]):

Definition 2. Let $\mathcal{J} = (G, H, I)$ be a context. A sequence $(g_0, m_0, g_1, m_1, \dots, g_{r-1}, m_{r-1}, g_r)$, where $g_i \in G$ for $i \in \{0, \dots, r\}$, $m_j \in H$ for $j \in \{0, \dots, r-1\}$ and $g_j I m_j, g_{j+1} I m_j$ for all $j \in \{0, \dots, r-1\}$, is called a *path* between elements g_0 and g_r . In a similar way we can define a *path* between two elements of H .

A positive integer r is said to be a *length of a path* between elements g_0, g_r . We suppose that the path (g, m, g) has a length 0. If a path between two elements of G exists, then we say that they are *joinable*. The context \mathcal{J} is said to be *irreducible* if every two elements of G are joinable. The minimal length of all paths between elements $g, h \in G$ we call a *distance* of these elements and denote by $v(g, h)$. The maximal distance of any two elements of G in an irreducible context \mathcal{J} is said to be a *span* of G and denoted by $d(G)$. Similarly for the set H .

We will investigate the contexts of independent sets (their joinability, distances, irreducibility, spans) associated to the lattice $(\mathcal{P}(M), \subseteq)$ where $\mathcal{P}(M)$ denotes the power set of a non-empty set M . Thus $(\mathcal{P}(M), \subseteq)$ is the complete (boolean) lattice of all subsets of M .

Let us denote by $\mathcal{M} = \{\{a\} \mid a \in M\} \subseteq \mathcal{P}(M)$ the set of all atoms of $(\mathcal{P}(M), \subseteq)$. This set (and every its subset) is obviously join-independent.

Further we put $\mathcal{N} = \{s_a \mid a \in M\}$ where $s_a = \vee \mathcal{M}_{\{a\}} = \vee (\mathcal{M} \setminus \{\{a\}\})$. Then \mathcal{N} is the set of all coatoms of $(\mathcal{P}(M), \subseteq)$ and it is meet-independent (also every its subset).

In what follows, $\mathcal{J}_L^p = (L_j^p, L_m^p, I^p)$ denotes the context of the p -element independent sets associated to the lattice $(\mathcal{P}(M), \subseteq)$, where M is a non-empty set and p is any cardinal number.

Proposition 8. *The following statements are equivalent:*

1. $|M| < p$,
2. $L_j^p = \emptyset$.

Proof. $1 \implies 2$: Let $A = \{A_i \mid i \in J\} \in L_j^p$ where $A_i \subseteq M$ and $|J| = p$, $|M| < p$. If we put $J_i := J \setminus \{i\}$, then $A_i \not\subseteq \bigcup_{j \in J_i} A_j$ for all $i \in J$. This implies $(A_i \setminus \bigcup_{j \in J_i} A_j) = A^i \neq \emptyset$. For each $a \in A^i$ we have $a \notin A_j$ for all $j \in J_i$. Then we can make a choice $M' = \{a^i \in A^i \mid i \in J\} \subseteq M$ and $\alpha : a^i \mapsto i$ is a one-to-one mapping of the subset M' of M onto J . However, this is a contradiction to $|M| < p$.

$2 \implies 1$: Let us assume that $L_j^p = \emptyset$ and $p \leq |M|$. Then there exists a subset $\mathcal{M}' \subseteq \mathcal{M}$ such that $|\mathcal{M}'| = p$. Since every subset of \mathcal{M} is join-independent, we get $\mathcal{M}' \in L_j^p$ and $L_j^p \neq \emptyset$. Thus $|M| < p$. ■

Proposition 9. *Let p be a finite cardinal number. Then the following statements are equivalent:*

1. $|M| = p$,
2. $L_j^p = \{\mathcal{M}\}$.

Proof. $1 \implies 2$: Let $A = \{A_i \mid i \in J\} \in L_j^p$, $A_i \subseteq M$ and $|J| = p = |M|$. Then $A_i \not\subseteq \bigcup_{j \in J_i} A_j$ for all $i \in J$, where $J_i = J \setminus \{i\}$ again. Hence $A_i \setminus \bigcup_{j \in J_i} A_j = A^i \neq \emptyset$ for all $i \in J$.

Assume that $x \in A^r \cap A^s$ for some $r, s \in J, r \neq s$. Then $x \in A^r, x \notin A_j$ for all $j \in J, j \neq r$. Thus $x \notin A^s$ which is a contradiction to $x \in A^s \subseteq A_s$. We have obtained $A^i \cap A^j = \emptyset$ for all $i, j \in J, i \neq j$.

If we make a choice $M' = \{a^i \in A^i \mid i \in J\} \subseteq M$, then $\alpha : a^i \mapsto i$ is a bijection of M' onto J and because of $|M| = |J|$ we have $M' = M$. Therefore $|A^i| = 1$ for all $i \in J$. Let $A^t = \{a\}$ for a certain $t \in J$. Then $a \in A_t$. If $b \in A_t, b \neq a$, then at the same time $b \in A^u$ for a certain $u \neq t$.

This yields $b \notin A_t$ which is a contradiction. Hence, $|A_i| = 1$ for all $i \in J$. We have proved that $A_i = \{a_i\}$ for all $i \in J$. It means that the only p -element join-independent set is \mathcal{M} .

$2 \implies 1$: According to the previous proposition, $p \leq |M|$. Every p -element set of atoms $\{\{a_i\} \mid i \in I\} \subseteq \mathcal{P}(M)$, $|I| = p$, is join-independent. If $p < |M|$, then there exist at least two distinct p -element sets of atoms. Thus $|L_j^p| > 1$. ■

Example. If $|M| = 3$, then $|L_j^2| = 9$ and $|L_j^3| = 1$. If $|M| = 4$, then $|L_j^2| = 55$, $|L_j^3| = 26$ and $|L_j^4| = 1$.

Proposition 10. *The set $\{A_i \mid i \in J\}$ is join-independent in $(\mathcal{P}(M), \subseteq)$ if and only if the set $\{M \setminus A_i \mid i \in J\}$ is meet-independent in $(\mathcal{P}(M), \subseteq)$.*

Proof. For all $i \in J$ we put $J_i = J \setminus \{i\}$. Then it is easy to see that

$$A_i \not\subseteq \bigcup_{j \in J_i} A_j \Leftrightarrow M \setminus \bigcup_{j \in J_i} A_j \not\subseteq M \setminus A_i \Leftrightarrow \bigcap_{j \in J_i} (M \setminus A_j) \not\subseteq M \setminus A_i.$$

■

Remark 4. It follows from Propositions 8 – 10 that $p > |M|$ if and only if $\mathcal{J}_L^p = (\emptyset, \emptyset, \emptyset)$, and $p = |M|$ if and only if $|L_j^p| = |L_m^p| = 1$. Also in the case $p < |M|$ we get $|L_j^p| = |L_m^p|$.

Proposition 11. *Let $A, B \in L_j^p$, $A = \{A_i \mid i \in J\}$, $B = \{B_i \mid i \in J\}$, $|J| = p$. If we denote $C = \{M \setminus A_i \mid i \in J\}$, $D = \{M \setminus B_i \mid i \in J\}$, then $v(A, B) = 1$ if and only if $v(C, D) = 1$.*

Proof. Assume that $v(A, B) = 1$. Then there exists $\bar{A} \in L_m^p$ such that $AI^p \bar{A}$, $BI^p \bar{A}$. Let us put $\bar{A} = \{\bar{A}_i \mid i \in J\}$ and $J_i = J \setminus \{i\}$. Under a suitable enumeration we get $\bar{A}_i \in X^A(A_i) \cap X^B(B_i)$ for all $i \in J$. Thus $\bigcup_{j \in J_i} A_j \subseteq \bar{A}_i$, $A_i \not\subseteq \bar{A}_i$ and $\bigcup_{j \in J_i} B_j \subseteq \bar{A}_i$, $B_i \not\subseteq \bar{A}_i$ for all $i \in J$. Let us put $\bar{C}_i = M \setminus \bar{A}_i$. Then we have $\bar{C}_i = M \setminus \bar{A}_i \subseteq M \setminus \bigcup_{j \in J_i} A_j$, $\bar{C}_i \not\subseteq M \setminus A_i$, $\bar{C}_i \subseteq M \setminus \bigcup_{j \in J_i} B_j$, $\bar{C}_i \not\subseteq M \setminus B_i$. This yields $\bar{C}_i \in Y^C(M \setminus A_i)$ and $\bar{C}_i \in Y^D(M \setminus B_i)$, thus $\bar{C}_i \in Y^C(M \setminus A_i) \cap Y^D(M \setminus B_i)$ for all $i \in J$. If we denote $\bar{C} = \{\bar{C}_i \mid i \in J\}$, then $\bar{C}I^p C$, $\bar{C}I^p D$ and $v(C, D) = 1$. Similarly for the converse assertion. ■

Proposition 12. *The sets $A = \{A_i \mid i \in J\}$, $B = \{B_i \mid i \in J\} \in L_j^p$ are joinable in \mathcal{J}_L^p if and only if the sets $C = \{M \setminus A_i \mid i \in J\}$, $D = \{M \setminus B_i \mid i \in J\}$ are joinable in \mathcal{J}_L^p .*

Proof. The sets $A, B \in L_j^p$ are joinable if and only if there exist sets $A'_1, \dots, A'_r \in L_j^p$, $A''_1, \dots, A''_{r+1} \in L_m^p$ such that $AI^p A''_1, A'_1 I^p A''_1, A'_1 I^p A''_2, A'_2 I^p A''_2, \dots, A'_r I^p A''_r, A'_r I^p A''_{r+1}, BI^p A''_{r+1}$. Thus, $v(A, A'_1) = v(A'_1, A'_2) = \dots = v(A'_r, B) = 1$. It follows from propositions 10 and 11 that there exist meet-independent sets $\bar{A}'_1, \bar{A}'_2, \dots, \bar{A}'_r$ such that $v(C, \bar{A}'_1) = v(\bar{A}'_1, \bar{A}'_2) = \dots = v(\bar{A}'_r, D) = 1$. Hence, the sets C, D are joinable. Similarly for the converse assertion. ■

Remark 5. If $A \subseteq \mathcal{M}$ (the subset of atoms), then for $\{a\} \in A$ we will write just $X^A(a), A_a, U(a)$ etc. instead of (more correct) $X^A(\{a\}), A_{\{a\}}, U(\{a\})$ etc. Then $X^A(a) = U(\vee A_a) \setminus U(a)$ and hence, $Y_a \in X^A(a)$ if and only if $A_a \subseteq Y_a, a \notin Y_a$.

Proposition 13. If $A, B \subseteq \mathcal{M}$, $A \neq B, |A| = |B| = p$, then $v(A, B) = 1$.

Proof. Let us denote $C = A \cap B$. There exists a bijective mapping $\varphi: A \rightarrow B$ such that $\varphi(c) = c$ for all $c \in C$. Further we put $Y_a = A_a \cup B_{\varphi(a)}$ for all $a \in A$. If $a \in C$, then $a = \varphi(a)$ and $a \notin A_a, B_a$. Thus $a \notin Y_a$. If $a \notin C$, then $a \notin B$ and $a \notin Y_a$. Similarly, $\varphi(a) \notin A$ and $\varphi(a) \notin Y_a$. It follows that $Y_a \in X^A(a) \cap X^B(\varphi(a))$. If we put $Y = \{Y_a \mid a \in A\}$, then $A \rightarrow Y : a \mapsto Y_a$ and $B \rightarrow Y : \varphi(a) \mapsto Y_a$ are norming mappings. Thus, $AI^p Y, BI^p Y$ and $v(A, B) = 1$. ■

Proposition 14. If $A, B \subseteq \mathcal{N}$, $A \neq B, |A| = |B| = p$, then $v(A, B) = 1$.

Proof. Dual to the previous one. ■

Theorem 1. Let \mathcal{J}_L^p be a context of independent sets associated to the complete lattice $(\mathcal{P}(M), \subseteq)$, where M is a non-empty set and p is a cardinal number with the property $3 \leq p < |M|$. Then \mathcal{J}_L^p is irreducible and (the span) $d(L_j^p) = 2$.

Proof. Consider join-independent sets $A = \{A_i \mid i \in J\}$, $B = \{B_i \mid i \in J\}$, where $A_i, B_i \subseteq M$ for all $i \in J$, $|J| = p$. For each $i \in J$, we put $J_i = J \setminus \{i\}$ and $A^i = \bigcup_{j \in J_i} A_j$. Then $Y \in X^A(A_i)$ if and only if $A^i \subseteq Y, A_i \not\subseteq Y$. Since A is join-independent, we have $A_i \not\subseteq A^i$ for all $i \in J$. It follows that there always exists an element $a_i \in A_i$ such that $a_i \notin A^i$. Then $A^i \subseteq M_{a_i} = s_{a_i}$. From $a_i \notin s_{a_i}$, we get $A_i \not\subseteq s_{a_i}$ and hence $s_{a_i} \in X^A(A_i)$. We can make a choice $Y_1 = \{s_{a_i} \mid i \in J\}$. The set $Y_1 \subseteq \mathcal{N}$ is meet-independent and $AI^p Y_1$. In a similar way, we can proceed in the case of the set B and we obtain

BI^pY_2 for a certain set $Y_2 \subseteq \mathcal{N}$. According to Proposition 14, there exists a set $C \subseteq \mathcal{M}$, $|C| = p$ such that CI^pY_1 , CI^pY_2 . Thus $v(A, B) \leq 2$.

It remains to find join-independent sets $A = \{A_i \mid i \in J\}$, $B = \{B_i \mid i \in J\}$, $|J| = p$, such that $v(A, B) = 2$. We determine them in the following way: Consider three distinct elements $a, b, c \in M$. Let us put $A_1 = \{a, b\} = B_1$, $A_2 = \{a, c\}$, $B_2 = \{b, c\}$ and $A_i = B_i = \{x_i\}$ for the other sets where $x_i \in M$ are pairwise distinct elements not equal to a, b, c . Moreover, we denote $C = \{a, b, c\}$ and $X = \{x_i \mid i \in J'\}$.

It is easy to verify that the sets A, B defined above are join-independent. Obviously, $X^A(x_i) = X^B(x_i)$ for all $i \in J'$ and $X^A(A_2) = X^B(B_2)$. It is also clear that $Y \subseteq X^A(A_1)$ if and only if $\{a, c\} \cup X \subseteq Y$, $A_1 \not\subseteq Y$, and $Y \subseteq X^B(B_1)$ if and only if $\{b, c\} \cup X \subseteq Y$, $B_1 \not\subseteq Y$. Let $Y \in X^A(A_1) \cap X^B(B_1)$. Then $C \cup X \subseteq Y$ which is a contradiction to $A_1, B_1 \subseteq Y$. Therefore, there is no meet-independent set Z such that AI^pZ , BI^pZ . Thus $v(A, B) = 2$. ■

Remark 6. Dually we can prove that also every two meet-independent sets are joinable and $d(L_m^p) = 2$.

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