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BALANCED CONGRUENCES*

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Abstract

Let $\mathcal{V}$ be a variety with two distinct nullary operations 0 and 1. An algebra $\mathfrak{A} \in \mathcal{V}$ is called balanced if for each $\Phi, \Psi \in Con(\mathfrak{A})$, we have $[0]\Phi = [0]\Psi$ if and only if $[1]\Phi = [1]\Psi$. The variety $\mathcal{V}$ is called balanced if every $\mathfrak{A} \in \mathcal{V}$ is balanced. In this paper, balanced varieties are characterized by a Mal'cev condition (Theorem 3). Furthermore, some special results are given for varieties of bounded lattices.

Keywords: balanced congruence, balanced algebra, balanced variety, Mal'cev condition.

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1. Balanced congruences on bounded lattices

Let $\mathfrak{L} = (L; \vee, \wedge, 0, 1)$ be a bounded lattice with least element 0 and greatest element 1. We study congruences on $\mathfrak{L}$ such that the 0-class determines the 1-class and vice versa.

Let $M \subseteq L$ and $a, b \in L$. Denote by $\Theta(M)$ the least congruence on $\mathfrak{L}$ containing the relation $M \times M$ and by $\Theta(a, b)$ the principal congruence on $\mathfrak{L}$ generated by (a, b), i.e. $\Theta(a, b) = \Theta(M)$ for $M = \{a, b\}$. We will use the following result of G. Grätzer and E. T. Schmidt, see [3]:

Proposition 1. *Let $\mathfrak{L}$ be a distributive lattice and $a, b, c, d \in L$ with $c \leq d$. Then $a\Theta(c, d)b$ if and only if $a \wedge c = b \wedge c$ and $a \vee d = b \vee d$.*

Here and in the following, we write $a\Phi b$ instead of $(a, b) \in \Phi$, for any $\Phi \in Con(\mathfrak{L})$ and $a, b \in L$.

From now on, every lattice under consideration is bounded, i.e. it has 0 and 1.

Definition 1. Let $\mathfrak{L}$ be a lattice with 0 and 1 and $\Phi \in Con(\mathfrak{L})$. Put $I = [0]\Phi$ and $F = [1]\Phi$. We say that Φ is *balanced* if

$$[0]\Phi = [0]\Theta(F) \text{ and } [1]\Phi = [1]\Theta(I).$$

Example 1. By Corollary 2.1 in [2], every complemented lattice is locally regular at 0 (in the sense of [1]), i.e. for every $\Phi, \Psi \in Con(\mathfrak{L})$, if $[a]\Phi = [a]\Psi$ for some $a \in L$, then $[0]\Phi = [0]\Psi$. Dually, it is locally regular at 1, i.e. $[a]\Phi = [a]\Psi$ implies $[1]\Phi = [1]\Psi$. Thus every congruence on every complemented lattice is balanced.

Definition 2. Let $\mathfrak{L}$ be a lattice with 0 and 1. We say that $\mathfrak{L}$ is a *d-lattice* if for each $a, b, c, d \in L$ the following holds:

$$a\Theta(c, 1)0 \;\Rightarrow\; a \wedge c = 0$$
$$b\Theta(d, 0)1 \;\Rightarrow\; b \vee d = 1.$$

Example 2. The lattice $\mathfrak{N}_5$ is a d-lattice. Of course, $\mathfrak{N}_5$ has just five congruences: the identical one ω, the full square $N_5 \times N_5$ and three nontrivial ones $\Theta_1, \Theta_2, \Theta_3$ defined by the following partitions (see Figure 1):

$$\Theta_1 \;\ldots\; \{0, b, c\},\ \{a, 1\}$$
$$\Theta_2 \;\ldots\; \{0\},\ \{a\},\ \{b, c\},\ \{1\}$$
$$\Theta_3 \;\ldots\; \{0, a\},\ \{b, c, 1\}.$$

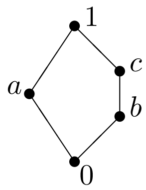

Figure 1

It is an easy exercise to verify our condition for $\Theta_1, \Theta_2, \Theta_3$; for ω and $N_5 \times N_5$ it is trivial (since $\omega = \Theta(0,0) = \Theta(1,1)$ and $N_5 \times N_5 = \Theta(0,1)$).

Lemma 1. *Every bounded distributive lattice is a d-lattice.*

The proof follows immediately by Proposition 1 putting $b = 0$ and $d = 1$ in the first case and $a = 1$ and $c = 0$ in the second one. ∎

Lemma 2. *Let* $\mathfrak{L}$ *be a bounded lattice and* $\Phi \in Con(\mathfrak{L})$. *Take* $S = \{a \in L : c \wedge a = 0$ *for some* $c \in [1]\Phi\}$. *Then* $S \subseteq [0]\Phi$.

Proof. Let $a \in S$. Then $a \wedge c = 0$ for some $c \in [1]\Phi$. Hence, $c\Phi 1$ thus also $(c \vee a)\Phi(1 \vee a) = 1$, i.e. $c \vee a \in [1]\Phi$. It yields $c\Phi(c \vee a)$ and $0 = (c \wedge a)\Phi((c \vee a) \wedge a) = a$ proving $a \in [0]\Phi$. ∎

Theorem 1. *Let* $\mathfrak{L}$ *be a d-lattice and* $\Phi \in Con(\mathfrak{L})$. *Then* Φ *is balanced if and only if*

$$[0]\Phi = \{a \in L : \ c \wedge a = 0 \ \textit{ for some } \ c \in [1]\Phi\} \ \textit{and}$$
$$[1]\Phi = \{b \in L : \ d \vee b = 1 \ \textit{ for some } \ d \in [0]\Phi\}.$$

Proof. Let $\mathfrak{L}$ be a d-lattice and let $\Phi \in Con(\mathfrak{L})$ be balanced. Define a set $S = \{a \in L : \ c \wedge a = 0$ for some $c \in [1]\Phi\}$. By Lemma 2 we have $S \subseteq [0]\Phi$. Conversely, let $a \in [0]\Phi$. Since Φ is balanced, we have $0\Theta(F)a$ for $F = [1]\Phi$. Applying the fact that $Con(\mathfrak{L})$ is a compactly generated lattice, there exist elements $c_1, \ldots, c_k \in F$ such that $0[\Theta(1, c_1) \vee \ldots \vee \Theta(1, c_k)]a$. Set $c = c_1 \wedge \ldots \wedge c_k$. Then $c \leq c_i \leq 1$ whence $\Theta(1, c_i) \subseteq \Theta(1, c)$, i.e. $0\Theta(1, c)a$. Since $\mathfrak{L}$ is a d-lattice, it implies $0 = c \wedge a$. Since $c \in F$, we have $a \in S$, i.e. $[0]\Phi \subseteq S$. We have shown $[0]\Phi = S$. Dually it can be shown that $[1]\Phi = \{b \in L : \ d \vee b = 1$ for some $d \in [0]\Phi\}$.

Conversely, let $\Phi \in Con(\mathfrak{L})$, and suppose that $[0]\Phi$, $[1]\Phi$ are described as in Theorem 1. Set $I = [0]\Phi$, $F = [1]\Phi$ and $\Psi_1 = \Theta(F)$, $\Psi_2 = \Theta(I)$. Evidently, $[1]\Psi_1 = F$ and $[0]\Psi_2 = I$.

It gives immediately

$$[0]\Phi = \{a \in L : \ c \wedge a = 0 \ \text{ for some } \ c \in F\} = \\ = \{a \in L : \ c \wedge a = 0 \ \text{ for some } \ c \in [1]\Psi_1\}$$

thus $I \subseteq [0]\Psi_1$ by Lemma 2. However, both Φ and Ψ_1 have the 1-class F and Ψ_1 is the least congruence with this property, i.e. $\Psi_1 \subseteq \Phi$ whence $[0]\Psi_1 \subseteq I$. We have shown that $[0]\Phi = I = [0]\Theta(F)$.

Analogously one can prove $[1]\Phi = [1]\Theta(I)$, i.e. Φ is balanced. ■

Example 3. Consider the distributive lattice $\mathfrak{L}$ visualized in Figure 2. Consider the congruences Φ_1 and Φ_2 determined by the following partitions:

$$\Phi_1 \quad \ldots \quad \{0, a\}, \{p, q\}, \{c, 1\}$$
$$\Phi_2 \quad \ldots \quad \{0, p\}, \{a, q\}, \{c\}, \{1\}.$$

Then Φ_1 is balanced since

$$[0]\Phi_1 = \{0, a\} = \{x \in L : \ c \wedge x = 0\},$$
$$[1]\Phi_1 = \{c, 1\} = \{y \in L : \ a \vee y = 1\}.$$

On the contrary, Φ_2 is not balanced since $[1]\Phi_2 = \{1\} = [1]\omega$, whereas $[0]\Phi_2 \neq [0]\omega$.

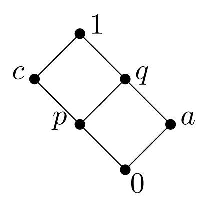

Figure 2

2. Balanced algebras with two nullary operations

Definition 3. Let τ be a type containing two distinct nullary operations denoted by 0 and 1 and let $\mathfrak{A}$ be an algebra of type τ. A congruence $\Phi \in Con(\mathfrak{A})$ is called *balanced* if

$$[0]\Phi = [0]\Theta(F) \text{ and } [1]\Phi = [1]\Theta(I)$$

where, as in Definition 1, $I = [0]\Phi, F = [1]\Phi$ and for $M \subseteq A, \Theta(M)$ denotes the least congruence on $\mathfrak{A}$ containing $M \times M$. The algebra $\mathfrak{A}$ is called *balanced* if each $\Phi \in Con(\mathfrak{A})$ is balanced. A variety $\mathcal{V}$ of type τ is called *balanced* if every $\mathfrak{A} \in \mathcal{V}$ has this property.

Lemma 3. *An algebra* $\mathfrak{A}$ *with nullary operations* 0 *and* 1 *is balanced if and only if for each* $\Phi, \Psi \in Con(\mathfrak{A})$ *the following condition holds:*

$$[0]\Phi = [0]\Psi \textit{ if and only if } [1]\Phi = [1]\Psi.$$

Proof. Of course, if Φ and Ψ are balanced and $[0]\Phi = [0]\Psi = I$, then $[1]\Phi = [1]\Theta(I) = [1]\Psi$ and vice versa. Conversely, suppose that $[0]\Phi = [0]\Psi$ if and only if $[1]\Phi = [1]\Psi$ for each $\Phi, \Psi \in Con(\mathfrak{A})$. Then $[0]\Phi = I = [0]\Theta(I)$, thus also $[1]\Phi = [1]\Theta(I)$ and similarly, $[0]\Phi = [0]\Theta(F)$ with $F = [1]\Phi$, i.e. every congruence on $\mathfrak{A}$ is balanced. So $\mathfrak{A}$ is balanced. ■

Example 4. The lattice $\mathfrak{N}_5$ (see Figure 1) is balanced, since for Θ_2 (of Example 2), we have $[0]\Theta_2 = [0]\omega$ and $[1]\Theta_2 = [1]\omega$. We have $\Theta_2 \neq \omega$, so it is a nontrivial example.

The consequence of Example 1 is that every complemented lattice is balanced (this yields another proof that $\mathfrak{N}_5$ is balanced). However, this condition is not necessary. For example, every simple lattice is balanced but it need not be complemented, see e.g. Figure 3.

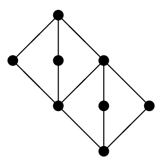

Figure 3

Problem. If $\mathfrak{L}$ is a bounded distributive lattice (or a d-lattice), is it true that $\mathfrak{L}$ is balanced if and only if $\mathfrak{L}$ is complemented?

Lemma 4. *Let $\mathfrak{A}$ be an algebra with nullary operations 0 and 1. If $\mathfrak{A}$ is balanced then for each $\Phi \in Con(\mathfrak{A})$ the following property holds:*

(S) $\quad [0]\Phi$ *is a singleton if and only if* $[1]\Phi$ *is a singleton .*

Proof. Suppose, e.g., that $[1]\Phi = \{1\}$. Then $[1]\Phi = [1]\omega$ for the identical congruence $\omega \in Con(\mathfrak{A})$, thus, by Lemma 3, also $[0]\Phi = [0]\omega = \{0\}$. The converse can be shown analogously. ■

For varieties, the converse statement is also valid:

Theorem 2. *Let $\mathcal{V}$ be a variety of type τ containing two distinct nullary operations 0 and 1. $\mathcal{V}$ is balanced if and only if for each $\mathfrak{A} \in \mathcal{V}$ and every $\Phi \in Con(\mathfrak{A})$ property (S) holds.*

Proof. Let $\mathcal{V}$ satisfy condition (S), let $\mathfrak{A} \in \mathcal{V}$ and $\Phi, \Psi \in Con(\mathfrak{A})$. Suppose $[1]\Phi = [1]\Psi$. Then clearly also $[1]\Phi = [1]\Phi \wedge \Psi$, so we can assume $\Psi \subseteq \Phi$ without loss of generality. Consider the factor algebra $\mathfrak{A}/\Psi$ and the factor congruence $\Phi/\Psi \in Con(\mathfrak{A}/\Psi)$. Since $[1]\Psi = [1]\Phi$, the class of Φ/Ψ containing the element $[1]\Psi \in \mathfrak{A}/\Psi$ is a singleton. Thus, by (S), also the class $[[0]\Psi]\Phi/\Psi$ is a singleton whence $[0]\Psi = [0]\Phi$.

Analogously we can show that $[0]\Phi = [0]\Psi \Rightarrow [1]\Phi = [1]\Psi$, thus $\mathfrak{A}$ and hence also $\mathcal{V}$ is balanced, by Lemma 3. The converse assertion follows directly by Lemma 4. ■

3. A characterization of balanced varieties

The following Theorem characterizes balanced varieties by a Mal'cev condition.

Theorem 3. *Let $\mathcal{V}$ be a variety of type τ which contains two distinct nullary operations 0 and 1. The following conditions are equivalent:*

1. *$\mathcal{V}$ is balanced.*
2. *There exist – for some m, n, k and h – unary terms $p_1, \ldots, p_m, q_1, \ldots, q_n$, $(2m+1)$-ary terms $r_1, \ldots, r_k$ and $(2n+1)$-ary terms $s_1, \ldots, s_h$ such that the following identities hold in $\mathcal{V}$:*

$p_1(0) = \ldots = p_m(0) = 1,\ q_1(1) = \ldots = q_n(1) = 0,$ *and*
$x = r_1(p_1(x), \ldots, p_m(x), 1, \ldots, 1, x),$
$r_i(1, \ldots, 1, p_1(x), \ldots, p_m(x), x) = r_{i+1}(p_1(x), \ldots, p_m(x), 1, \ldots, 1, x),$
$i = 1, \ldots, k-1,$
$0 = r_k(1, \ldots, 1, p_1(x), \ldots, p_m(x), x),$ *and*
$x = s_1(q_1(x), \ldots, q_n(x), 0, \ldots, 0, x),$
$s_j(0, \ldots, 0, q_1(x), \ldots, q_n(x), x) = s_{j+1}(q_1(x), \ldots, q_n(x), 0, \ldots, 0, x),$
$j = 1, \ldots, h-1,$
$1 = s_h(0, \ldots, 0, q_1(x), \ldots, q_n(x), x).$

3. *There exist – for some m, n – unary terms $p_1, \ldots, p_m, q_1, \ldots, q_n$ and 3-ary terms $r_1, \ldots, r_m, s_1, \ldots, s_n$ such that the following identities hold in $\mathcal{V}$:*
   $p_1(0) = \ldots = p_m(0) = 1, q_1(1) = \ldots = q_n(1) = 0,$ *and*
   $x = r_1(p_1(x), 1, x),$
   $r_i(1, p_i(x), x) = r_{i+1}(p_{i+1}(x), 1, x), i = 1, \ldots, m-1,$
   $0 = r_m(1, p_m(x), x),$ *and*
   $x = s_1(q_1(x), 0, x),$
   $s_j(0, q_j(x), x) = s_{j+1}(q_{j+1}(x), 0, x), j = 1, \ldots, n-1,$
   $1 = s_n(0, q_n(x), x).$
4. *There exist – for some m, n – unary terms $p_1, \ldots, p_m, q_1, \ldots, q_n$ such that*

$$[p_1(x) = 1 \& \ldots \& p_m(x) = 1] \quad \Leftrightarrow \quad x = 0,$$
$$[q_1(x) = 0 \& \ldots \& q_n(x) = 0] \quad \Leftrightarrow \quad x = 1.$$

Proof. (1) $\Rightarrow$ (2): Let $\mathfrak{A} = \mathfrak{F}_{\mathcal{V}}(x)$ be a free algebra of $\mathcal{V}$ with one free generator x, let $\Psi = \Theta(x, 0) \in Con(\mathfrak{A})$ and $B = [1]\Psi$. Set $\Phi = \Theta(B)$, i.e., Φ is the least congruence on $\mathfrak{A}$ having the 1-class equal to B. Then $[1]\Psi = [1]\Phi$ thus, by (1), also $[0]\Psi = [0]\Phi$, i.e., $0\Phi x$. As the lattice $Con(\mathfrak{A})$ is compactly generated, there exist $b_1, \ldots, b_m \in B$ such that

$$(*) \qquad x[\Theta(1, b_1) \vee \ldots \vee \Theta(1, b_m)]0.$$

Since $b_i \in \mathfrak{F}_{\mathcal{V}}(x)$, there exist unary terms $p_i(x)$ such that $b_i = p_i(x)$ for $i = 1, \ldots, m$. Since $b_i \in [1]\Theta(x, 0)$, it implies immediately

$$p_i(0) = 1 \text{ for } i = 1, \ldots, m.$$

Applying the well-known Mal'cev scheme (see [4]) on $(*)$, we obtain

$$x = r_1(p_1(x), \ldots, p_m(x), 1, \ldots, 1, x),$$
$$r_j(1, \ldots, 1, p_1(x), \ldots, p_m(x), x) = r_{j+1}(p_1(x), \ldots, p_m(x), 1, \ldots, 1, x),$$
$$0 = r_k(1, \ldots, 1, p_1(x), \ldots, p_m(x), x)$$

for some $(2m+1)$-ary terms $r_1, \ldots, r_k$ and $j = 1, \ldots, k-1$.

If we set $\Psi = \Theta(x, 1)$ and $B = [0]\Psi$, then the constants 0 and 1 only interchange their roles and we obtain the remaining identities of (2) analogously.

(2) $\Rightarrow$ (3): For $i = 2, \ldots, m-1$ and $j = 1, \ldots, k$ put

$$r_{j1}(u, v, x) = r_j(u, p_2(x), \ldots, p_m(x), v, 1, \ldots, 1, x)$$
$$r_{ji}(u, v, x) = r_j(1, \ldots, 1, u, p_{i+1}(x), \ldots, p_m(x), p_1(x), \ldots, p_{i-1}(x), v, 1, \ldots, 1, x),$$

where there are $i-1$ nullary operations 1 at the beginning and $m-i$ nullary operations 1 after v, and

$$r_{jm}(u, v, x) = r_j(1, \ldots, 1, u, p_1(x), \ldots, p_{m-1}(x), v, x).$$

Furthermore, put $p_{ji}(x) = p_i(x)$ for $i = 1, \ldots, m$ and $j = 1, \ldots, k$.

Then we have for $i = 1, \ldots, m-1$ and $j = 1, \ldots, k$:

$$r_{ji}(1, p_{ji}(x), x) =$$
$$= r_j(1, \ldots, 1, 1, p_{i+1}(x), \ldots, p_m(x), p_1(x), \ldots, p_{i-1}(x), p_i(x), 1, \ldots, 1, x) =$$
$$= r_{j,i+1}(p_{j,i+1}(x), 1, x)$$

and for $j = 1, \ldots, k-1$:

$$r_{jm}(1, p_{jm}(x), x) = r_j(1, \ldots, 1, 1, p_1(x), \ldots, p_{m-1}(x), p_m(x), x) =$$
$$= r_{j+1}(p_1(x), \ldots, p_m(x), 1, \ldots, 1, x) =$$
$$= r_{j+1,1}(p_{j+1,1}(x), 1, x),$$

where the second identity follows from (2).

Furthermore we have, again by (2):

$$r_{11}(p_{11}(x), 1, x) = r_1(p_1(x), p_2(x), \ldots, p_m(x), 1, 1, \ldots, 1, x) = x,$$
$$r_{km}(1, p_{km}(x), x) = r_k(1, \ldots, 1, 1, p_1(x), \ldots, p_{m-1}(x), p_m(x), x) = 0.$$

By writing m instead of km, $r_1, \ldots, r_m$ instead of $r_{11}, \ldots, r_{km}$ and $p_1, \ldots, p_m$ instead of $p_{11}, \ldots, p_{km}$ (both in lexicographic order), we obtain the first part of (3). The second part of (3) can be shown analogously.

(3) $\Rightarrow$ (4): Take the same terms p_i, q_j as in (3). Then $p_i(0) = 1$, $q_j(1) = 0$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

If we suppose $p_i(x) = 1$ for $i = 1, \ldots, m$, then, by (3), we obtain

$$\begin{aligned} x &= r_1(p_1(x), 1, x) = r_1(1, 1, x) = \\ &= r_2(1, 1, x) = \ldots = r_k(1, 1, x) = \\ &= r_m(1, p_m(x), x) = 0, \end{aligned}$$

i.e., $x = 0$, thus we have the first implication of (4). The second one can be shown analogously.

(4) $\Rightarrow$ (1): Let $\mathfrak{A} \in \mathcal{V}$ and $\Phi \in Con(\mathfrak{A})$. Suppose, e.g., $[1]\Phi = \{1\}$ and take $b \in [0]\Phi$. Then $[b]\Phi = [0]\Phi$ and, by (4), we have in the factor algebra $\mathfrak{A}/\Phi$

$$[p_i(b)]\Phi = p_i([b]\Phi) = p_i([0]\Phi) = [1]\Phi = \{1\}$$

whence $p_i(b) = 1$ for $i = 1, \ldots, m$. Applying (4) again, we conclude $b = 0$, thus $[0]\Phi = \{0\}$. Analogously we can show

$$[0]\Phi = \{0\} \Rightarrow [1]\Phi = \{1\}.$$

By Theorem 2 , $\mathcal{V}$ is balanced. ■

Remark. The equivalence of (2) and (3) can also be proved directly since (3) $\Rightarrow$ (2) can be seen as follows:
Take the r_i in (3) and define the $(2m+1)$-ary terms $\bar{r}_i$ by

$$\bar{r}_i(x_1, \ldots, x_m, y_1, \ldots, y_m, x) = r_i(x_i, y_i, x), i = 1, \ldots, m,$$

then we have

$$\begin{aligned} &\bar{r}_i(p_1(x), \ldots, p_m(x), 1, \ldots, 1, x) = r_i(p_i(x), 1, x) \text{ and} \\ &\bar{r}_i(1, \ldots, 1, p_1(x), \ldots, p_m(x), x) = r_i(1, p_i(x), x), \end{aligned}$$

and we obtain the identities of (2) with $m = k$ and $\bar{r}_i$ instead of r_i. Similarly, we obtain the identities of (2) with $n = h$ and some $\bar{s}_i$ instead of s_i.

Example 5. The variety of all ortholattices $(L; \vee, \wedge, ^\perp, 0, 1)$ is balanced. We can take $m = 1 = n$ and $p_1(x) = q_1(x) = x^\perp$. Then $p_1(x) = 1$ if and only if $x = 0$ and $q_1(x) = 0$ if and only if $x = 1$.

Example 6. Every variety of double p-algebras is balanced (a double p-algebra is an algebra $(L;\vee,\wedge,^*,^+,0,1)$ of type $(2,2,1,1,0,0)$, where $(L;\vee,\wedge,0,1)$ is a bounded lattice, a^* is a pseudocomplement of $a\in L$ and a^+ is a dual pseudocomplement). We can take $m=1=n$ and $p_1(x)=x^*$, $q_1(x)=x^+$. Of course, $0^*=1$, $1^+=0$, and $x^*=1\Rightarrow x=0$, $x^+=0\Rightarrow x=1$.

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