

ON DISTRIBUTIVE TRICES

KIYOMITSU HORIUCHI

*Department of Information Science and Systems Engineering,
Faculty of Science and Engineering, Konan University
Okamoto, Higashinada, Kobe 658-8501, Japan*
e-mail: horiuchi@konan-u.ac.jp

AND

ANDREJA TEPAVČEVIĆ

*Institute of Mathematics Fac. of Sci., University of Novi Sad
Trg D. Obradovića 4, 21000 Novi Sad Yugoslavia*
e-mail: etepavce@EUnet.yu

Abstract

A triple-semilattice is an algebra with three binary operations, which is a semilattice in respect of each of them. A trice is a triple-semilattice, satisfying so called roundabout absorption laws. In this paper we investigate distributive trices. We prove that the only subdirectly irreducible distributive trices are the trivial one and a two element one. We also discuss finitely generated free distributive trices and prove that a free distributive trice with two generators has 18 elements.

Keywords and phrases: triple semilattice, trice, distributive trice.

2000 Mathematics Subject Classification: 06A12, 08B20, 08B26.

1. INTRODUCTION

An algebra $(T; \nearrow_1, \searrow_2, \downarrow_3)$ of a type with three binary operations is a *triple semilattice* if it is a semilattice in respect of each of the operations. We denote orders on T by

- (1) $a \leq_1 b$ if and only if $a \nearrow_1 b = b$,
- (2) $a \leq_2 b$ if and only if $a \searrow_2 b = b$,
- (3) $a \leq_3 b$ if and only if $a \downarrow_3 b = b$.

A triple semilattice T is a *trice* if it satisfies the *roundabout absorption laws*:

$$(4) \quad ((a \nearrow_1 b) \nwarrow_2 b) \downarrow_3 b = b,$$

$$(5) \quad ((a \nearrow_1 b) \downarrow_3 b) \nwarrow_2 b = b,$$

$$(6) \quad ((a \nwarrow_2 b) \nearrow_1 b) \downarrow_3 b = b,$$

$$(7) \quad ((a \nwarrow_2 b) \downarrow_3 b) \nearrow_1 b = b,$$

$$(8) \quad ((a \downarrow_3 b) \nearrow_1 b) \nwarrow_2 b = b,$$

and

$$(9) \quad ((a \downarrow_3 b) \nwarrow_2 b) \nearrow_1 b = b$$

for all $a, b \in T$.

Trices are introduced and investigated in [2] as a generalization of lattices.

A *distributive trice* is a trice satisfying the following six distributive laws:

$$(10) \quad a \nearrow_1 (b \nwarrow_2 c) = (a \nearrow_1 b) \nwarrow_2 (a \nearrow_1 c),$$

$$(11) \quad a \nwarrow_2 (b \nearrow_1 c) = (a \nwarrow_2 b) \nearrow_1 (a \nwarrow_2 c),$$

$$(12) \quad a \nearrow_1 (b \downarrow_3 c) = (a \nearrow_1 b) \downarrow_3 (a \nearrow_1 c),$$

$$(13) \quad a \downarrow_3 (b \nearrow_1 c) = (a \downarrow_3 b) \nearrow_1 (a \downarrow_3 c),$$

$$(14) \quad a \nwarrow_2 (b \downarrow_3 c) = (a \nwarrow_2 b) \downarrow_3 (a \nwarrow_2 c),$$

and

$$(15) \quad a \downarrow_3 (b \nwarrow_2 c) = (a \downarrow_3 b) \nwarrow_2 (a \downarrow_3 c)$$

for all $a, b, c \in T$.

2. SUBDIRECT DECOMPOSITION OF DISTRIBUTIVE TRICES

Lemma 1. *A triple semilattice T having all three semilattices as chains is a trice if and only if for all $x, y \in T$, there are \leq_i and \leq_j , for $i, j \in \{1, 2, 3\}$, such that $x \leq_i y$ and $y \leq_j x$.*

Proof. By contraposition, if for all orderings $x \leq_i y$ $i \in \{1, 2, 3\}$ is satisfied, than $x \nearrow_1 (x \nwarrow_2 (x \downarrow_3 y)) = y$, i.e., roundabout absorption law (9) is not satisfied. On the other hand, if, say, $x \leq_1 y$ and $y \leq_2 x$, then it is easy to prove that all roundabout absorption laws for x and y are satisfied. ■

Lemma 2. *Let $(T; \nearrow_1, \searrow_2, \downarrow_3)$ be a distributive trice. Let $x, y, t \in T$. If $x \nearrow_1 t = y \nearrow_1 t$, $x \searrow_2 t = y \searrow_2 t$ and $x \downarrow_3 t = y \downarrow_3 t$, then $x = y$.*

Proof. Using repeatedly the hypotheses, we have

$$\begin{aligned}
x &= x \nearrow_1 (x \searrow_2 (x \downarrow_3 t)) = x \nearrow_1 (x \searrow_2 (y \downarrow_3 t)) \\
&= x \nearrow_1 ((x \searrow_2 y) \downarrow_3 (x \searrow_2 t)) = x \nearrow_1 ((x \searrow_2 y) \downarrow_3 (y \searrow_2 t)) \\
&= x \nearrow_1 (y \searrow_2 (x \downarrow_3 t)) = x \nearrow_1 (y \searrow_2 (y \downarrow_3 t)) \\
&= (x \nearrow_1 y) \searrow_2 (x \nearrow_1 (y \downarrow_3 t)) = (x \nearrow_1 y) \searrow_2 ((x \nearrow_1 y) \downarrow_3 (x \nearrow_1 t)) \\
&= (x \nearrow_1 y) \searrow_2 ((x \nearrow_1 y) \downarrow_3 (y \nearrow_1 t)) = (x \nearrow_1 y) \searrow_2 (y \nearrow_1 (x \downarrow_3 t)) \\
&= (x \nearrow_1 y) \searrow_2 (y \nearrow_1 (y \downarrow_3 t)) = y \nearrow_1 (x \searrow_2 (y \downarrow_3 t)) \\
&= y \nearrow_1 ((x \searrow_2 y) \downarrow_3 (x \searrow_2 t)) = y \nearrow_1 ((x \searrow_2 y) \downarrow_3 (y \searrow_2 t)) \\
&= y \nearrow_1 (y \searrow_2 (x \downarrow_3 t)) = y \nearrow_1 (y \searrow_2 (y \downarrow_3 t)) \\
&= y.
\end{aligned}$$

■

Let $(T; \nearrow_1, \searrow_2, \downarrow_3)$ be a distributive trice, and let $p \in T$ be a fixed element. We define relations on T by

$$(16) \quad x \theta_1 y \text{ if and only if } x \nearrow_1 p = y \nearrow_1 p,$$

$$(17) \quad x \theta_2 y \text{ if and only if } x \searrow_2 p = y \searrow_2 p,$$

and

$$(18) \quad x \theta_3 y \text{ if and only if } x \downarrow_3 p = y \downarrow_3 p.$$

Lemma 3. *The relations θ_1 , θ_2 and θ_3 defined by (16)–(18) are congruences on the distributive trice.*

Proof. It is obvious that every θ_i , for $i \in \{1, 2, 3\}$ is an equivalence relation. Moreover, it is compatible with all operations. Let $x \theta_1 y$ and $z \theta_1 t$. Then $x \nearrow_1 p = y \nearrow_1 p$ and $z \nearrow_1 p = t \nearrow_1 p$. And then $(x \nearrow_1 p) \searrow_2 (z \nearrow_1 p) = (y \nearrow_1 p) \searrow_2 (t \nearrow_1 p)$. By distributivity, $(x \searrow_2 z) \nearrow_1 p = (y \searrow_2 t) \nearrow_1 p$, i.e., $(x \searrow_2 z) \theta_1 (y \searrow_2 t)$. Similarly, we get $(x \downarrow_3 z) \theta_1 (y \downarrow_3 t)$. Hence, θ_1 is a congruence on the trice. For θ_2 and θ_3 , we can prove it in a similar way. ■

Lemma 4. *The relation θ_i is the identity relation if and only if p is the bottom element in the $(T; \leq_i)$, for all $i \in \{1, 2, 3\}$.*

Proof. If p is the bottom element in (T, \leq_1) , then $p \leq_1 x$ for all $x \in T$. Hence, $x \theta_1 y$ if and only if $x = x \nearrow_1 p = y \nearrow_1 p = y$. That is, $\theta_1 = \Delta$.

On the other hand, if there exists $x \in T$ such that $\neg(p \leq_1 x)$, then $(p \nearrow_1 x) \neq x$. As $(p \nearrow_1 x) \nearrow_1 p = x \nearrow_1 p$, we get $(p \nearrow_1 x) \theta_1 x$. Then, $\theta_1 \neq \Delta$. For θ_2 and θ_3 , we can prove the statements in a similar way. ■

Lemma 5. *If p is not the bottom element of any of semilattices of the distributive trice T , then not all of θ_1 , θ_2 and θ_3 are equal.*

Proof. Suppose that all the congruences are equal. Let $x <_1 p$. Then, $x \theta_1 p$. As congruences are the same, $x \theta_2 p$ and $x \theta_3 p$. Hence $x \leq_2 p$ and $x \leq_3 p$, and thus $((x \nearrow_1 p) \frown_2 p) \downarrow_3 p = p$, and finally from our assumption we obtain $p = x$, a contradiction. ■

Lemma 6. *There are no subdirectly irreducible distributive trices with more than three elements.*

Proof. Suppose that T is a subdirectly irreducible distributive trice with four or more elements. Then, there is an element, say $p \in T$, which is not the bottom element in any of the semilattices. This element determines three congruences θ_1 , θ_2 and θ_3 , defined by formulas (16) – (18). Those relations are all distinct from the identity relation by Lemma 4, and at least two of them are not equal by Lemma 5. Using Lemma 2 we easily prove that

$$\theta_1 \cap \theta_2 \cap \theta_3 = \Delta.$$

By the well known theorem on congruence lattice of subdirectly irreducible algebras (see e.g. [1], p. 57. Thm. 8.4), we have that T is not subdirectly irreducible. ■

Lemma 7. *There are no subdirectly irreducible distributive trices with three elements.*

Proof. There is only one (up to the isomorphism and the order of operations) distributive trice with three elements $(T; \nearrow_1, \frown_2, \downarrow_3)$, diagrams of its semilattices given in Figure 1. It is not subdirectly irreducible. Indeed, congruences of this trice, besides Δ and ∇ , are $\{\{a, b\}, \{c\}\}$, and $\{\{a\}, \{b, c\}\}$, that is, congruence lattice is the four element boolean algebra. Thus, this trice is not subdirectly irreducible. ■

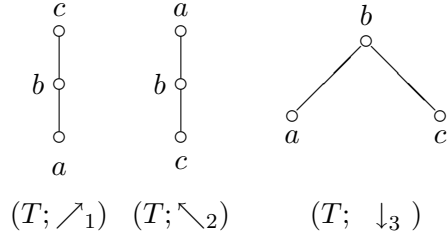


Figure 1

By lemmas 1 – 7, we have:

Theorem 1. *The only subdirectly irreducible distributive trices are, up to the isomorphism and the order of operations, the two element one, given in Figure 2, and the trivial one.*

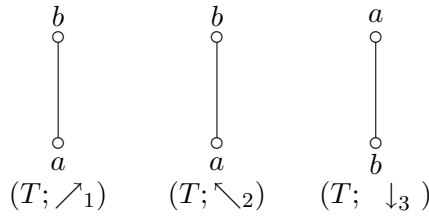


Figure 2

■

Theorem 2. *Every non-trivial distributive trice is isomorphic to a subdirect product of two element trices.*

Proof. This is a consequence of previous theorem and the Birkhoff theorem on subdirect products. ■

An obvious corollary is that every distributive trice is a subtrice of the direct product of two element trices.

Example 1. In the sequel, we give representation of the three element distributive trice in Figure 1, as a subdirect product of two element trices.

Proof. Let $T_1 = \{a, b\}$ and $T_2 = \{c, d\}$, with $a \leq_1 b$, $a \leq_2 b$, $b \leq_3 a$, $c \leq_1 d$, $d \leq_2 c$ and $d \leq_3 c$. The direct product has four elements $\{ac, bc, ad, bd\}$. The mentioned three element trice is isomorphic with the subtrice $\{ac, bc, bd\}$.

3. FREE DISTRIBUTIVE TRICES

In the sequel we consider free distributive trices.

Obviously, free distributive trice with one generator is the one element trivial trice. Now, consider n generators, x_1, \dots, x_n . Every element of a free distributive trice can be written in the form $F_1 \downarrow_3 F_2 \downarrow_3 \dots \downarrow_3 F_m$, where every $F_i (i \in \{1, 2, \dots, m\})$ is of the form: $g_1 \searrow_2 g_2 \searrow_2 \dots \searrow_2 g_k$, and every $g_j (j \in \{1, 2, \dots, k\})$ is of the form: $x_{i_1} \nearrow_1 x_{i_2} \nearrow_1 \dots \nearrow_1 x_{i_l}$, where all x_s appearing in the mentioned expression, are generators. We can easily prove, by using distributive laws, that every element of a free distributive trice have a representation of that form. And obviously, some elements have several different representations.

By the previous considerations, the following theorem is evident:

Theorem 3. *Every free distributive trice with a finite set of generators is finite.*

Proof. Let n be the number of generators. Let G be the set of all elements of the form: $x_{i_1} \nearrow_1 x_{i_2} \nearrow_1 \dots \nearrow_1 x_{i_l}$, where all x_s appearing in the mentioned expressions are generators. Then, the cardinality of G is not greater than $2^n - 1$. Let F be the set of all elements of the form: $g_1 \searrow_2 g_2 \searrow_2 \dots \searrow_2 g_k$, where $g_i \in G$, for all $i \in \{1, \dots, k\}$. Then, the cardinality of F is not greater than $2^{2^n - 1} - 1$. As every element of a free distributive trice can be written in the form $F_1 \downarrow_3 F_2 \downarrow_3 \dots \downarrow_3 F_m$, with $F_i \in F$, the order of free distributive trice with n generators is not greater than $2^{2^{2^n - 1} - 1} - 1$. There is some possibility of overlapping. But, this completes the proof. ■

Example 2. Free distributive trice with two generators has 18 elements.

We effectively construct a free distributive trice with two generators x and y . The notations in the sequel is taken from the proof of the previous theorem. Now, the set G is $\{x, y, x \nearrow_1 y\}$. From $x \searrow_2 y \searrow_2 (x \nearrow_1 y) = (x \searrow_2 y \searrow_2 x) \nearrow_1 (x \searrow_2 y \searrow_2 y) = x \searrow_2 y$, it follows that the set F is $\{x, y, x \nearrow_1 y, x \searrow_2 y, x \searrow_2 (x \nearrow_1 y), y \searrow_2 (x \nearrow_1 y)\}$. In a similar way, we can deduce that the free distributive trice with two generators has 18 elements.

All different elements of that trice are represented by the following terms:

- ① = x , ② = y , ③ = $x \nearrow_1 y$,
- ④ = $x \nwarrow_2 y = x \nwarrow_2 y \nwarrow_2 (x \nearrow_1 y)$, ⑤ = $x \nwarrow_2 (x \nearrow_1 y)$,
- ⑥ = $y \nwarrow_2 (x \nearrow_1 y)$, ⑦ = $x \downarrow_3 y$, ⑧ = $x \downarrow_3 (x \nearrow_1 y)$,
- ⑨ = $x \downarrow_3 (x \nwarrow_2 y)$, ⑩ = $y \downarrow_3 (x \nearrow_1 y)$, ⑪ = $y \downarrow_3 (x \nwarrow_2 y)$,
- ⑫ = $x \downarrow_3 (y \nwarrow_2 (x \nearrow_1 y))$, ⑬ = $y \downarrow_3 (x \nwarrow_2 (x \nearrow_1 y))$,
- ⑭ = $(x \nwarrow_2 y) \downarrow_3 (x \nearrow_1 y) = (x \nwarrow_2 (x \nearrow_1 y)) \downarrow_3 (y \nwarrow_2 (x \nearrow_1 y))$,
- ⑮ = $(x \nearrow_1 y) \downarrow_3 (x \nwarrow_2 (x \nearrow_1 y))$, ⑯ = $(x \nearrow_1 y) \downarrow_3 (y \nwarrow_2 (x \nearrow_1 y))$,
- ⑰ = $(x \nwarrow_2 y) \downarrow_3 (x \nwarrow_2 (x \nearrow_1 y))$, ⑱ = $(x \nwarrow_2 y) \downarrow_3 (y \nwarrow_2 (x \nearrow_1 y))$.

Diagrams of the free distributive trice with two generators are presented by Figures 3.1–3.3. The orders in each of the semilattices of the trice are represented by arrows. ③, ④ and ⑦ are the top elements in the orders \leq_1 , \leq_2 and \leq_3 , respectively.

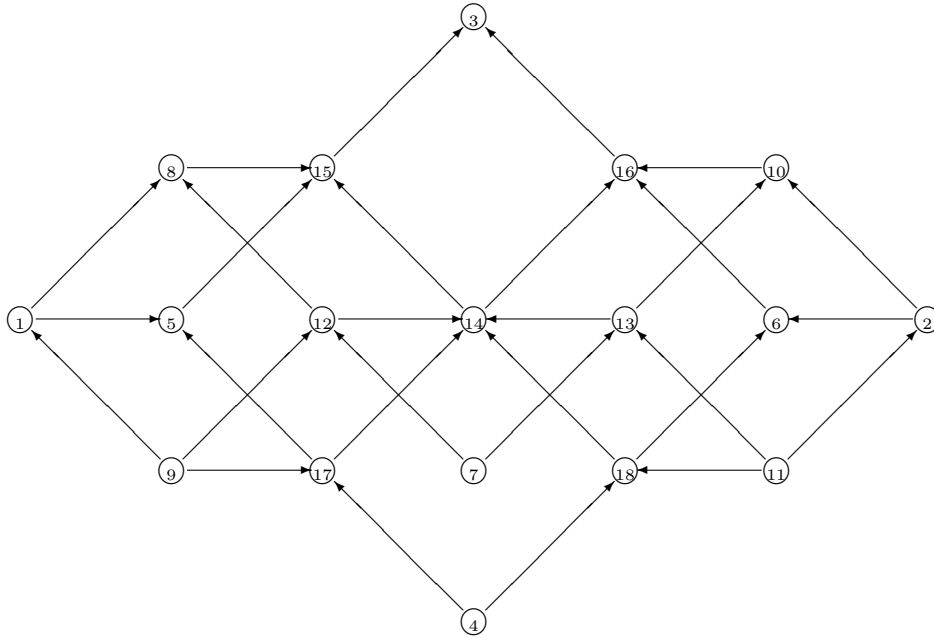


Figure 3 – 1

The order \leq_1

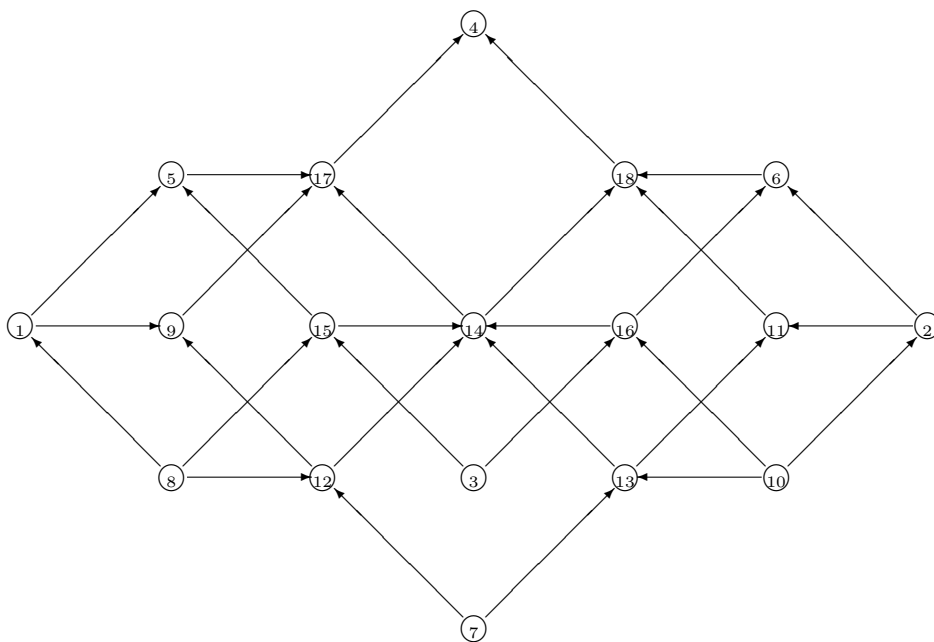


Figure 3 – 2

The order \leq_2

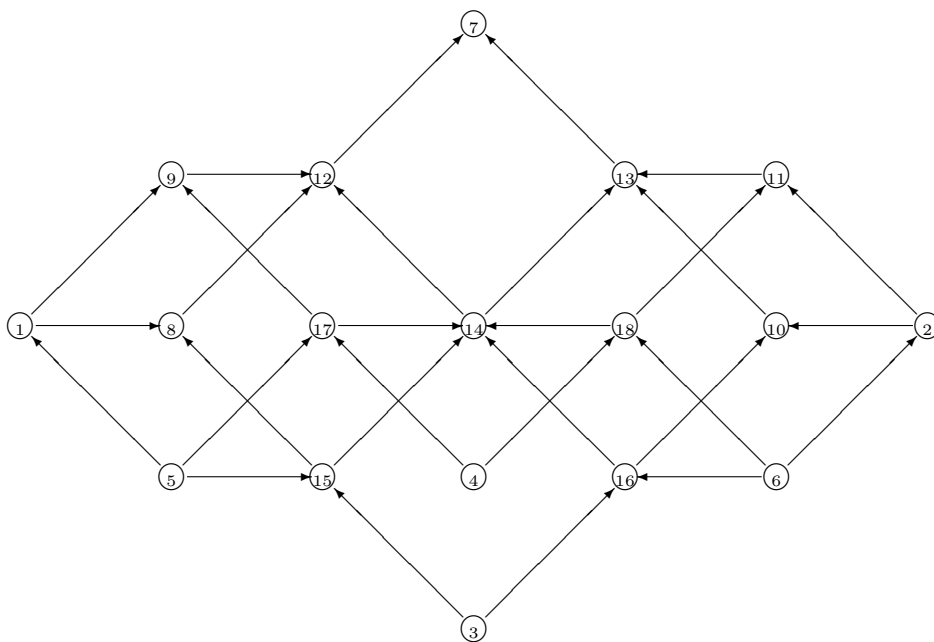


Figure 3 – 3

The order \leq_3

REFERENCES

- [1] S. Burris, and H.P. Sankappanavar, *A Course in Universal Algebra*, Springer-Verlag, New York 1981 (new, electronic version, 1999: is available at the address: www.thoralf.uwaterloo.ca).
- [2] K. Horiuchi, *Trice and Two delegates operation*, *Sci. Math.* **2** (1999), 373–384.
- [3] J.A. Kalman, *Subdirect decomposition of distributive quasi-lattices*, *Fund. Math.* **71** (1971), 161–163.

Received 17 May 1999
Revised 12 March 2001