Discussiones Mathematicae
General Algebra and Applications 20 (2000) 219–231

CONGRUENCES ON PSEUDOCOMPLEMENTED SEMILATTICES

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Abstract

It is known that congruence lattices of pseudocomplemented semilattices are pseudocomplemented [4]. Many interesting properties of congruences on pseudocomplemented semilattices were described by Sankappanavar in [4], [5], [6]. Except for other results he described congruence distributive pseudocomplemented semilattices [6] and he characterized pseudocomplemented semilattices whose congruence lattices are Stone, i.e. belong to the variety $\mathcal{B}_1$ [5].

In this paper we give a partial solution to a more general question: Under what condition on a pseudocomplemented semilattice its congruence lattice is element of the variety $\mathcal{B}_n$ $(n \geq 2)$?

In the last section we widen the Sankappanavar's result to obtain the description of pseudocomplemented semilattices with relative Stone congruence lattices. A partial solution of the description of pseudocomplemented semilattices with relative (L_n)-congruence lattices $(n \geq 2)$ is also given.

Keywords: pseudocomplemented semilattice, congruence lattice, p-algebra, Stone algebra, (relative) (L_n)-lattice.

1991 Mathematics Subject Classification: Primary 06D15, 06A99; Secondary 08A30

1. Preliminaries

A pseudocomplemented semilattice (PCS) is an algebra $S = \langle S; \wedge, ^*, 0\rangle$, where $\langle S; \wedge, 0\rangle$ is a $\wedge$-semilattice with 0 and * is the unary operation of pseudocomplementation defined by:

$$x \wedge a = 0 \text{ iff } x \leq a^*.$$

0^* is the largest element in S and is denoted by 1. An element $a \in S$ is called *closed* if $a = a^{**}$. The set of all closed elements of S is denoted $B(S)$. It is known that $\langle B(S); +, \wedge, ^*, 0, 1\rangle$ forms a Boolean algebra in which the operation of join is defined by $a+b = (a^* \wedge b^*)^*$. To denote the join of subset $A \subseteq B(S)$ of closed elements we will use the symbol $\sum A$.

An element $d \in S$ is called *dense* if $d^* = 0$. All dense elements form the set denoted $D(S)$ which is a filter in S.

The set of all congruences on PCS S is denoted $Con(S)$. It is known that $Con(S)$ is an algebraic pseudocomplemented lattice [4] with Δ and ∇ the least and the largest element, respectively.

For any pair $a, b \in S$ the symbol $\theta(a, b)$ denotes the *principal congruence relation generated* by a, b, i.e. the least congruence relation θ on S for which $(a, b) \in \theta$.

The congruence relation φ defined by:

$$(x, y) \in \varphi \text{ iff } x^* = y^*,$$

is called the *Glivenko congruence relation*.

For arbitrary filter $F \subseteq S$ we define binary relation $\hat{F}$:

$$(x, y) \in \hat{F} \text{ iff } x \wedge f = y \wedge f \text{ for some } f \in F.$$

Clearly $\hat{F}$ is a semilattice congruence relation on S. For arbitrary element $f \in S$ the interval $[0, f] \subseteq S$ is a PCS such that the pseudocomplement $a^*_{[0,f]}$ is equal to $a^* \wedge f$. It follows that $\hat{F}$ is compatible also with the operation of pseudocomplementation and $\hat{F} \in Con(S)$. Similarly for arbitrary element $a \in S$ we define binary relation $\hat{a}$ by

$$(x, y) \in \hat{a} \text{ iff } x \wedge a = y \wedge a.$$

Again $\hat{a} \in Con(S)$. One can easily verify that $\hat{a} = \theta(a, 1)$ for arbitrary $a \in S$.

The following two facts were proved by Sankappanavar in [4] and [6].

Lemma 1.1. *Let S be a* PCS. *If $\psi \in Con(S)$ then $\psi = ([1]\psi)^\wedge \vee (\psi \wedge \varphi)$.* ■

Lemma 1.2. *Let S be a* PCS. *The following statesments are equivalent:*

(1) *$Con(S)$ is distributive,*

(2) *S satisfies:*

(D) $$\forall x \forall y (x < y^{**} \Rightarrow x \leq y \text{ or } y \leq x),$$

(3) *S satisfies:*

(D_w) $$\forall x \forall y (x^* = y^* \Rightarrow x \leq y \text{ or } y \leq x)$$

and

(U′) $$\forall x \forall y ((x = x^{**} \text{ and } x < y^{**}) \Rightarrow x < y),$$

(4) *$Con(S)$ is modular.* ■

One can easily verify the next auxiliary lemma.

Lemma 1.3. *Let S be a* PCS *satisfying* (D)*. Let $a, b \in S$ be such that $a < b$ and $a^* = b^*$. Then*

(i) $\theta(a, b) = [a, b] \times [a, b] \cup \Delta$;

(ii) $[1]\theta^*(a, b) = [b, 1]$. ■

A (distributive) *p-algebra* is an algebra $L = \langle L; \vee, \wedge, ^*, 0, 1\rangle$ where $\langle L; \vee, \wedge, 0, 1\rangle$ is a bounded (distributive) lattice and * is the unary operation of pseudocomplementation. learly the congruence lattice of any congruence distributive PCS is a distributive p-algebra.

The class $\mathcal{B}_\omega$ of all distributive p-algebras is equational. K.B. Lee proved in [3] that the lattice of all equational subclasses of $\mathcal{B}_\omega$ is a chain

$$\mathcal{B}_{-1} \subset \mathcal{B}_0 \subset \mathcal{B}_1 \subset \ldots \subset \mathcal{B}_n \subset \ldots \subset \mathcal{B}_\omega$$

of type $\omega + 1$, where $\mathcal{B}_{-1}, \mathcal{B}_0$ and $\mathcal{B}_1$ denote the classes of all trivial, Boolean and Stone algebras, respectively. Moreover, he proved that for $n \geq 1$, $L \in \mathcal{B}_n$ if and only if L satisfies the identity

(L_n) $$(x_1 \wedge x_2 \wedge \ldots \wedge x_n)^* \vee (x_1^* \wedge x_2 \wedge \ldots \wedge x_n)^* \vee \ldots \vee (x_1 \wedge x_2 \wedge \ldots \wedge x_n^*)^* = 1.$$

Definition 1.4 ([2]; Definition 1). Let L be a distributive p-algebra and $n \geq 1$. L is said to be an (L_n)-*lattice* if $L \in \mathcal{B}_n$.

2. Pseudocomplemented semilattices with (L_n)-congruence lattices

In [5] H.P. Sankappanavar gave a description of those PCS S whose congruence lattice $Con(S)$ is Stone, i.e. satisfies (L_1). The aim of this paper is to continue in this direction and investigate the cases for which $Con(S)$ satisfies (L_n) for $n \geq 2$.

Theorem 2.1. *Let S be a* PCS. *If $Con(S) \in \mathcal{B}_n$ $(n \geq 1)$, then*

(C_n) $\forall x_i$ $(x_i \neq x_i^{**}$ $(i = 1, \ldots, n+1)$ *and* $x_i \neq x_j$ $(i \neq j)) \Rightarrow \bigwedge_{i=1}^{n+1} x_i = 0$.

Proof. For $n = 1$, the claim was proved by H.P. Sankappanavar in Theorem 3.2 of [5]. Assume that $n \geq 2$. Let $x_1, x_2, \ldots, x_{n+1} \in S$ be such that $x_i \neq x_i^{**}$ $(i = 1, 2, \ldots, n+1)$ and $x_i \neq x_j$ $(i \neq j)$. Suppose that $w = \bigwedge_{i=1}^{n+1} x_i > 0$.

Without loss of generality we can divide elements $x_1, x_2, \ldots, x_{n+1}$ into k disjoint groups $(1 \leq k \leq n+1)$:

$$\{x_{11}, x_{12}, \ldots, x_{1m_1}\}, \{x_{21}, x_{22}, \ldots, x_{2m_2}\}, \ldots, \{x_{k1}, x_{k2} \ldots, x_{km_k}\}$$

such that $m_1 + m_2 + \ldots + m_k = n+1$ and

$$x_{i1} < x_{i2} < \ldots < x_{im_i} < x_{i1}^{**} \quad (i = 1, \ldots, k).$$

Let us denote

$$\begin{aligned}
&\tau_i = \theta(x_{1i}, x_{1i+1}), \quad i = 1, 2, \ldots, m_1 - 1\\
&\tau_{m_1} = \theta(x_{1m_1}, x_{11}^{**}),\\
&\tau_{m_1+j} = \theta(x_{2j}, x_{2j+1}), \quad j = 1, 2, \ldots, m_2 - 1\\
&\tau_{m_1+m_2} = \theta(x_{2m_2}, x_{21}^{**}),\\
&\ldots\\
&\tau_{m_1+m_2+\ldots+m_{k-1}+l} = \theta(x_{kl}, x_{kl+1}), \quad l = 1, 2 \ldots m_k - 1\\
&\tau_{m_1+m_2+\ldots+m_k} = \tau_{n+1} = \theta(x_{km_k}, x_{k1}^{**}).
\end{aligned}$$

Let $\theta_1 = \bigvee_{j=2}^{n+1} \tau_j$ and $\theta_i = \bigvee_{j=1}^{i-1} \tau_j \vee \bigvee_{j=i+1}^{n+1} \tau_j$, $i = 2, 3, \ldots, n$.

From Lemma 1.3 follows that $\theta_i^* \supseteq \tau_i$, $i = 1, 2, \ldots, n$.

Therefore, we have

$$\begin{array}{lll}
\theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_n \supseteq \tau_{n+1} & \text{and} & (\theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_n)^* \subseteq \tau_{n+1}^*; \\
\theta_1^* \wedge \theta_2 \wedge \ldots \wedge \theta_n \supseteq \tau_1 & \text{and} & (\theta_1^* \wedge \theta_2 \wedge \ldots \wedge \theta_n)^* \subseteq \tau_1^*; \\
 & \ldots & \\
\theta_1 \wedge \ldots \wedge \theta_i^* \wedge \ldots \wedge \theta_n \supseteq \tau_i & \text{and} & (\theta_1 \wedge \ldots \wedge \theta_i^* \wedge \ldots \wedge \theta_n)^* \subseteq \tau_i^*; \\
 & \ldots & \\
\theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_n^* \supseteq \tau_n & \text{and} & (\theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_n^*)^* \subseteq \tau_n^*.
\end{array}$$

From our assumption that $Con(S) \in \mathcal{B}_n$, we obtain

$$\tau_{n+1}^* \vee \tau_1^* \vee \tau_2^* \vee \ldots \vee \tau_n^* = \bigvee_{i=1}^{n+1} \tau_i^* = \nabla.$$

It implies that there exists a sequence $a_0 = 1, a_1, a_2, \ldots, a_m = 0 \subseteq S$ such that $a_i \equiv a_{i+1}(\alpha_{j(i)})$, $(i = 0, 1, \ldots, m-1)$, where $\alpha_{j(i)} \in \{\tau_k^* : \ k = 1, 2, \ldots, n+1\}$.

From Lemma 1.3, we obtain

$$\begin{array}{l}
[1]\tau_1^* \subseteq [x_{12}, 1] \subseteq [\bigwedge_{i=1}^{n+1} x_i, 1] = [w, 1], \\
[1]\tau_2^* \subseteq [x_{13}, 1] \subseteq [w, 1], \\
\ldots \\
[1]\tau_{m_1+m_2+\ldots+m_r+j}^* \subseteq [x_{r+1j+1}, 1] \subseteq [w, 1] \ (j = 1, 2, \ldots, m_{r+1} - 1), \\
\ldots \\
[1]\tau_{n+1}^* \subseteq [x_{k1}^{**}, 1] \subseteq [w, 1].
\end{array}$$

Clearly $a_1 \geq w$ and $a_{m-1}^* \geq w$, since $1 = a_0 \equiv a_1(\alpha_{j(0)})$ and $a_{m-1}^* \equiv 1(\alpha_{j(m-1)})$, $\alpha_{j(m-1)} \in \{\tau_k^* : \ k = 1, 2, \ldots, n+1\}$. If we meet elements $a_1, a_2, \ldots, a_{m-1}$ with the element a_{m-1}^*, we obtain a new sequence $b_1 = a_1 \wedge a_{m-1}^*, b_2 = a_2 \wedge a_{m-1}^*, \ldots, b_{m-2} = a_{m-2} \wedge a_{m-1}^*, b_{m-1} = 0$ such that $b_i \equiv b_{i+1}(\alpha_{j(i)})$, $(i = 1, 2, \ldots, m-2)$ and $\alpha_{j(i)} \in \{\tau_k^* : \ k = 1, 2, \ldots, n+1\}$. Again $b_1 = a_1 \wedge a_{m-1}^* \geq w$ and $b_{m-2}^* \geq w$. Repeating the previous step $m-2$ times we obtain $y \equiv 0(\alpha_{j(0)})$, $\alpha_{j(0)} \in \{\tau_k^* : \ k = 1, 2, \ldots, n+1\}$ such that $y \geq w$. Since $y^* \equiv 1(\alpha_{j(0)})$, $y^* \geq w$. Therefore, $w \leq y \wedge y^* = 0$ which is a contradiction with our assumption that $w = \bigwedge_{i=1}^{n+1} x_i > 0$. ■

Corollary 2.2. *Let S be a* PCS *such that $Con(S) \in \mathcal{B}_n$ $(n \geq 1)$. Then $|[a]\varphi| \leq n+1$ for arbitrary $a \in S$.* ■

Definition 2.3. Let S be a PCS. We say that S is an (S_n)-*semilattice* $(n \geq 1)$ iff S satisfies (C_n) and S satisfies (D). In other words, S is an (S_n)-semilattice if and only if S is a congruence distributive pseudocomplemented semilattice which satisfies the condition (C_n).

In the next we will often deal with non-closed elements. We find it useful to introduce now a few notations.

$N(S) = \{n \in S: \ n \text{ is non} - \text{closed }\}$, i.e.

$N(S) = \{n \in S: \ n \neq n^{**}\}$;

$N^{**}(S) = \{n^{**}: \ n \in N(S)\}$;

$C(S) = \{c \in S : c \wedge n = 0; \ \forall n \in N(S)\}$;

$C^*(S) = \{c^*: \ c \in C(S)\}$.

One can easily verify that $C(S)$ is an ideal in $B(S)$ and $0 \in C(S)$. Moreover, if $c \in C(S)$ and $n \in N(S)$, then $c \wedge n^{**} = (c \wedge n)^{**} = 0$. It follows that $C(S)$ can be defined equivalently as $C(S) = \{c \in S: \ c \wedge n^{**} = 0; \ \forall n \in N(S)\}$. If there is no danger of confusion, we will write N, N^{**}, C and C^* instead of $N(S), N^{**}(S), C(S)$ and $C^*(S)$, respectively.

Definition 2.4. Let S be a PCS and $\psi \in Con(S)$. Then

$N_\psi = \{n \in N: \ \theta(n, n^{**}) \wedge \psi \neq \Delta\}$,

$N_\psi^{**} = \{n^{**}: \ n \in N_\psi\}$.

Clearly C_ψ is an ideal in $B(S)$, $N_\psi = N_{\psi \wedge \varphi}$ and $N_\varphi = N$.

3. Properties of congruences on (S_n)-semilattices

The following lemmas were inspired by [5]. The next lemma is obvious.

Lemma 3.1. *Let S be a* PCS. *Then*

$$\varphi = \bigvee \{\theta(n, n^{**}): \ n \in N(S)\}.$$

For arbitrary $A \subseteq S$ the symbol A^u denotes the set of all upper bounds of A.

Lemma 3.2. *Let S be a* PCS. *Then $(N^{**})^u = C^*$.*

Proof. Let $n \in N$ and $c \in C$ be arbitrary. Then $c \wedge n^{**} = 0$. Therefore, $n^{**} \leq c^*$ and $C^* \subseteq (N^{**})^u$. Take arbitrary $y \in (N^{**})^u$. Clearly $y \in B(S)$. It means that $y = y^{**} \geq n^{**}$ for arbitrary $n \in N$. Thus $y^* \leq n^*$ and $y^* \wedge n = y^* \wedge n^{**} = 0$. It follows that $y^* \in C$ and since y is a closed element $y \in C^*$. ∎

Lemma 3.3. *Let S be a* PCS *satisfying* (D) *and $X \subseteq N(S) = N$. Then $N_{((X^{**})^u)^\wedge} \subseteq N \setminus X$.*

Proof. Suppose that $n \in X \cap N_{((X^{**})^u)^\wedge}$. Then there exist $n \leq n_1 < m_1 \leq n^{**}$ such that $n_1 \wedge f = m_1 \wedge f$ and $f \in (X^{**})^u$. Since $n \in X$, it follows that $f \geq n^{**}$. Thus $n_1 \wedge f = n_1 = m_1 = m_1 \wedge f$ contrary to our assumption $n_1 < m_1$. Therefore, $N_{((X^{**})^u)^\wedge} \cap X = \emptyset$ and $N_{((X^{**})^u)^\wedge} \subseteq N \setminus X$. ∎

Lemma 3.4. *Let S be a* PCS *satisfying* (D) *and $\beta \in Con(S)$ be such that $\beta \subseteq \varphi$. Then $((N_\beta^{**})^u)^\wedge \subseteq \beta^*$.*

Proof. Let $(x, y) \in \beta \wedge ((N_\beta^{**})^u)^\wedge$. Without loss of generality we can assume that $x < y \leq x^{**}$. Then $x \wedge f = y \wedge f$ for some $f \in (N_\beta^{**})^u$. Since $(x, y) \in \beta$, we obtain that $\theta(x, x^{**}) \wedge \beta \neq \Delta$ and $x \in N_\beta$. It implies that $f \geq x^{**} \geq y > x$ and $x \wedge f = x = y = y \wedge f$ contrary to our assumption $x < y$. So, we can conclude that $((N_\beta^{**})^u)^\wedge \subseteq \beta^*$. ∎

Corollary 3.5. *Let S be a* PCS *satisfying* (D). *Then $\varphi^* = ((N^{**})^u)^\wedge = (C^*)^\wedge$.*

Lemma 3.6. *Let S be an (S_n)-semilattice $(n \geq 1)$. Let $\psi \in Con(S)$ be such that $|[1]\psi \cap N| \geq n$. Then $\psi^* = \Delta$.*

Proof. Two cases can occur: $|[1]\psi \cap N| \geq n+1$ or $|[1]\psi \cap N| = n$. In the first case $\psi = \nabla$ since S is an (S_n)-semilattice. Thus $\psi^* = \Delta$.

In the second case we first claim that $\varphi \subseteq \psi$. If $N \subseteq [1]\psi$ then it is true. Assume that $N \subsetneq [1]\psi$. Let $[1]\psi \cap N = \{n_i : \ i = 1, \dots, n\}$. Let $s \in N \setminus [1]\psi$. Since $\bigwedge_{i=1}^n n_i \equiv 1(\psi)$ and S is an (S_n)-semilattice, we obtain that $s \wedge \bigwedge_{i=1}^n n_i = 0 \equiv s(\psi)$. Therefore, $s \equiv s^{**}(\psi)$ for arbitrary $s \in N$ and $\varphi \subseteq \psi$.

To complete the proof it suffices to show that also $\varphi^* \subseteq \psi$. Let $f \in (N^{**})^u$. Then $f \geq n_i^{**}$ for any $n_i \in [1]\psi \cap N$. It implies that $(N^{**})^u \subseteq [1]\psi$. Thus $\varphi^* = ((N^{**})^u)^\wedge \subseteq ([1]\psi)^\wedge \subseteq \psi$. Hence $\varphi \vee \varphi^* \subseteq \psi$. Therefore, we obtain $\psi^* \subseteq (\varphi \vee \varphi^*)^* = \varphi^* \wedge \varphi^{**} = \Delta$ proving the lemma. ∎

Definition 3.7 Let S be a PCS satisfying (D) and $A \subseteq C$. Then we define

$$d_C(A) = \{c \in C : c \wedge a = 0,\ a \in A\}.$$

Lemma 3.8. *Let S be a* PCS *satisfying* (D) *and $I \subseteq C$ be an ideal in $B(S)$. Then $(N^{**} \cup I \cup d_C(I))^u = \{1\}$.*

Proof. Let $f \in (N^{**} \cup I \cup d_C(I))^u$. Then

$$\begin{array}{lll} f \geq n^{**}\ (n^{**} \in N^{**}) & \text{and} & f^* \wedge n^{**} = 0; \\ f \geq a\ (a \in I) & \text{and} & f^* \wedge a = 0; \\ f \geq c\ (c \in d_C(I)) & \text{and} & f^* \wedge c = 0. \end{array}$$

From $f^* \wedge n^{**} = 0$ follows $f^* \in C$. Since $f^* \wedge a = 0$ for all $a \in I$, it follows $f^* \in d_C(I)$. Since $f^* \wedge c = 0$ for all $c \in d_C(I)$, we obtain that also $f^* \wedge f^* = f^* = 0$. Hence, f is a dense element. $f \in (N^{**})^u$ implies that f is closed. So we can conclude $f = 1$ proving the lemma. ■

By taking $I = \{0\}$, we immediately obtain

Corollary 3.9. *Let S be a* PCS *satisfying* (D). *Then $\{N^{**} \cup C\}^u = \{1\}$.* ■

Lemma 3.10. *Let S be a* PCS *satisfying* (D) *and $F \subseteq S$ be a Boolean filter, i.e. $F \subseteq B(S)$. Then $F \subseteq ((N^{**} \setminus N^{**}_{\hat{F}}) \cup d_C(C_{\hat{F}}))^u$.*

Proof. Let $f \in F$ be such that $f \notin (N^{**} \setminus N^{**}_{\hat{F}})^u$. Thus $f \ngeq n^{**}$ for some $n^{**} \in N^{**} \setminus N^{**}_{\hat{F}}$. Then $f \wedge n^{**} < n^{**}$ and, since $Con(S)$ is distributive, two possibilities may occure.

First suppose that $f \wedge n^{**} \leq n < n^{**}$. Since $f \equiv 1(\hat{F})$, $f \wedge n^{**} \equiv n^{**}(\hat{F})$, we obtain that $n \equiv n^{**}(\hat{F})$. Hence, $\theta(n, n^{**}) \wedge \hat{F} \neq \Delta$. Therefore, $n \in N_{\hat{F}}$, $n^{**} \in N^{**}_{\hat{F}}$ contrary to assumption $n^{**} \in N^{**} \setminus N^{**}_{\hat{F}}$. Now suppose that $n \leq f \wedge n^{**} < n^{**}$. Since $f \equiv 1(\hat{F})$, $f \wedge n^{**} \equiv n^{**}(\hat{F})$, we again obtain that $\theta(n, n^{**}) \wedge \hat{F} \neq \Delta$. Therefore, $n \in N_{\hat{F}}$, $n^{**} \in N^{**}_{\hat{F}}$ contrary to assumption $n^{**} \in N^{**} \setminus N^{**}_{\hat{F}}$. Thus $F \subseteq (N^{**} \setminus N^{**}_{\hat{F}})^u$.

Let $f \in F$ and $y \in d_C(C_{\hat{F}})$. Since $f^* \wedge f = 0 \wedge f$, we have $f^* \equiv 0(\hat{F})$ and also $f^* \wedge y \equiv 0(\hat{F})$. Thus, $f^* \wedge y \in C_{\hat{F}}$. From this, we get $(f^* \wedge y) \wedge y = f^* \wedge y = 0$. Hence, $y \leq f^{**} = f$ proving that $F \subseteq d_C(C_{\hat{F}})^u$. So we can conclude that $F \subseteq ((N^{**} \setminus N^{**}_{\hat{F}}) \cup d_C(C_{\hat{F}}))^u$. ■

Lemma 3.11. *Let S be a* PCS *satisfying* (D) *and let $F \subseteq S$ be a Boolean filter. Then $((N^{**}_{\hat{F}} \cup C_{\hat{F}})^u)^\wedge \subseteq (\hat{F})^*$.*

Proof. Let $(x,y) \in \hat{F} \wedge ((N^{**}_{\hat{F}} \cup C_{\hat{F}})^u)^\wedge$, $x < y$. It means that $x \wedge f = y \wedge f$ for some $f \in F$ and $x \wedge h = y \wedge h$ for some $h \in (N^{**}_{\hat{F}} \cup C_{\hat{F}})^u$. Therefore, $x^{**} \wedge f = y^{**} \wedge f$ and $x^{**} \wedge h^{**} = y^{**} \wedge h^{**}$. Since $x^{**}, y^{**}, f, h^{**} \in B(S)$, it follows that $x^{**} \wedge (f + h^{**}) = y^{**} \wedge (f + h^{**})$. Since $f \in F \subseteq ((N^{**} \setminus N^{**}_{\hat{F}}) \cup d_C(C_{\hat{F}}))^u$ and $h^{**} \in (N^{**}_{\hat{F}} \cup C_{\hat{F}})^u$, from the two previous lemmas we obtain that $f + h^{**} \in \{(N^{**} \setminus N^{**}_{\hat{F}}) \cup d_C(C_{\hat{F}}) \cup N^{**}_{\hat{F}} \cup C_{\hat{F}}\}^u = (N^{**} \cup C_{\hat{F}} \cup d_C(C_{\hat{F}}))^u = \{1\}$. Thus, we see that $x^{**} = y^{**}$. Since $(x,y) \in \hat{F}$, $x < y$ and $x^* = y^*$ we obtain that $\theta(x, x^{**}) \wedge \hat{F} \neq \Delta$ and $x \in N_{\hat{F}}$. Therefore, $h \geq x^{**} \geq y > x$ and $x \wedge h = x = y = y \wedge h$ which is a contradiction with our assumption $x < y$. Thus the lemma is proved. ∎

Theorem 3.12. *Let S be an (S_n)-semilattice $(n \geq 1)$ such that $B(S)$ is a complete Boolean algebra. Then $Con(S)$ is an (L_n)-lattice.*

Proof. For $n = 1$ the claim follows from [5] (see Theorem 3.27). Assume that $n \geq 2$. Let $\theta_1, \theta_2, \ldots, \theta_n$ be arbitrary elements of $Con(S)$. For the sake of simplicity let us denote

$$
\begin{aligned}
\alpha_0 &= \theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_n,\\
\alpha_1 &= \theta_1^* \wedge \theta_2 \wedge \ldots \wedge \theta_n,\\
&\ldots\\
\alpha_i &= \theta_1 \wedge \ldots \wedge \theta_i^* \wedge \ldots \wedge \theta_n,\\
&\ldots\\
\alpha_n &= \theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_n^*.
\end{aligned}
$$

We want to prove that $\alpha_0^* \vee \alpha_1^* \vee \ldots \vee \alpha_n^* = \nabla$. From Lemma 1.1, follows that $\alpha_i^* = (\alpha_i \wedge \varphi)^* \wedge (([1]\alpha_i)^\wedge)^*$ $(i = 0, 1, \ldots, n)$. Three possibilities may occur:

(1) $[1]\alpha_i \cap N \neq \emptyset$ for $i = 0, 1, \ldots, n$;

(2) $[1]\alpha_i \subseteq B(S)$ for $i = 0, 1, \ldots, n$;

(3) There exist $I, J \subseteq \{0, 1, \ldots, n\}$ such that $I \neq \emptyset \neq J$, $I \cap J = \emptyset$, $I \cup J = \{0, 1, \ldots, n\}$ and $[1]\alpha_i \subseteq B(S)$ for $i \in I$ and $[1]\alpha_j \cap N \neq \emptyset$ for $j \in J$.

Ad (1): Suppose that $n_i \in [1]\alpha_i \cap N$ $(i = 0, 1, \ldots, n)$. It means that $\theta(n_i, 1) \subseteq \alpha_i$. Since $\alpha_i \wedge \alpha_j = \Delta$ for $i, j \in \{0, 1, \ldots, n\}$, $i \neq j$, we obtain that $\theta(n_i, 1) \subseteq \alpha_j^*$ for arbitrary $j \neq i$. It follows that

$$\alpha_0^* \vee \alpha_1^* \vee \ldots \vee \alpha_n^* \supseteq \bigvee_{i=0}^{n} \theta(n_i, 1).$$

Let $\bigvee_{i=0}^n \theta(n_i, 1) = \theta$. Then $n_i \equiv 1(\theta)$ $(i = 0, 1, \ldots, n)$ and therefore $\bigwedge_{i=0}^n n_i \equiv 1(\theta)$. Since α_i $(i = 0, 1, \ldots, n)$ are pairwise disjoint, congruences n_i $(i = 0, 1, \ldots, n)$ are pairwise different nonclosed elements. From the assumption that S is an (S_n)-semilattice we obtain $\bigwedge_{i=0}^n n_i = 0 \equiv 1(\theta)$ hence $\alpha_0^* \vee \alpha_1^* \vee \ldots \vee \alpha_n^* = \nabla$.

Ad (2): Suppose that $[1]\alpha_i \subseteq B(S)$ for $i = 0, 1, \ldots, n$. From Lemma 3.4 and Lemma 3.11 follows that $\alpha_i^* \supseteq ((N^{**}_{\alpha_i \wedge \varphi})^u)^\wedge \wedge ((N^{**}_{([1]\alpha_i)^\wedge} \cup C_{([1]\alpha_i)^\wedge})^u)^\wedge$, $i = 0, 1, \ldots, n$. Let $\sum N^{**}_{\alpha_i \wedge \varphi} = a_i$, $\sum N^{**}_{([1]\alpha_i)^\wedge} = b_i$, $\sum C_{([1]\alpha_i)^\wedge} = c_i$, $(i = 0, 1, \ldots, n)$. Since $([1]\alpha_i)^\wedge \subseteq \alpha_i$ and $N^{**}_{\alpha_i \wedge \varphi} = N^{**}_{\alpha_i}$, we have $a_i = \sum N^{**}_{\alpha_i} \geq \sum N^{**}_{([1]\alpha_i)^\wedge} = b_i$ $(i = 0, 1, \ldots, n)$. Hence, $\alpha_i^* \supseteq \hat{a}_i \wedge (b_i + c_i)^\wedge = \theta(a_i, 1) \wedge \theta(b_i + c_i, 1) \supseteq \theta(a_i + b_i + c_i, 1) = \theta(a_i + c_i, 1)$ $(i = 0, 1, \ldots, n)$. Therefore, we have $\bigvee_{i=0}^n \alpha_i^* \supseteq \bigvee_{i=0}^n \theta(a_i + c_i, 1) = \theta(\bigwedge_{i=0}^n (a_i + c_i), 1)$. We claim that $a_i \wedge c_j = 0$ for arbitrary $i, j \in \{0, 1, \ldots, n\}$.

From the assumption that $B(S)$ is a complete Boolean algebra, it follows that $B(S)$ satisfies the join infinite distributive identity and its dual meet infinite distributive identity. Let $\sum N^{**} = m$. Since $c \wedge n^{**} = 0$ for arbitrary $c \in C, n \in N$, we obtain that $m = \sum N^{**} \leq c^*$ and therefore $c \leq m^*$ for arbitrary $c \in C$. Thus $\sum C \leq m^*$. It follows that $a_i \wedge c_j = \sum N^{**}_{\alpha_i} \wedge \sum C_{([1]\alpha_j)^\wedge} \leq \sum N^{**} \wedge \sum C \leq m \wedge m^* = 0$. Thus we obtain that $\bigwedge_{i=0}^n (a_i + c_i) = \bigwedge_{i=0}^n a_i + \bigwedge_{j=0}^n c_j$.

We claim that $\bigwedge_{j=0}^n c_j = 0$. Take arbitrary $i, j \in \{0, 1, \ldots, n\}$ such that $i \neq j$. Then $c_i \wedge c_j = \sum C_{([1]\alpha_i)^\wedge} \wedge \sum C_{([1]\alpha_j)^\wedge} = \sum \{d \wedge e : d \in C_{([1]\alpha_i)^\wedge}$ and $e \in C_{([1]\alpha_j)^\wedge}\}$. Since $([1]\alpha_i)^\wedge \wedge ([1]\alpha_j)^\wedge = \Delta$, we have $d \wedge e = 0$ for arbitrary $d \in C_{([1]\alpha_i)^\wedge}$ and $e \in C_{([1]\alpha_j)^\wedge}$. Hence, $c_i \wedge c_j = 0$.

Next we will prove that $\bigwedge_{i=0}^n a_i = 0$. Using the fact that $B(S)$ satisfies both the join and meet infinite distributive identities we obtain that $\bigwedge_{i=0}^n a_i = \bigwedge_{i=0}^n \sum N^{**}_{\alpha_i} = \sum \{\bigwedge_{i=0}^n n_i^{**} : n_i^{**} \in N^{**}_{\alpha_i}\}$. Take arbitrary $(n+1)$-tuple $(n_i^{**} : n_i^{**} \in N^{**}_{\alpha_i}, i = 0, 1, \ldots, n)$. Clearly some elements n_i^{**} $(i = 0, 1, \ldots, n)$ may coincide. Suppose that $n_i^{**} = m^{**}$ for $i \in I \subseteq \{0, 1, \ldots, n\}$. It means that there exist elements r_i, s_i such that $r_i < s_i$, $r_i^{**} = s_i^{**} = m^{**}$ and $(r_i, s_i) \in \alpha_i$ for $i \in I$. Since α_i are pairwise disjoin congruences, it follows that $\theta(r_i, s_i)$ $(i \in I)$ are also pairwise disjoint congruences. Thus $|I| \leq |[m^{**}]\varphi \cap N| \leq n$.

From the previous consideration follows that $\bigwedge_{i=0}^{n} n_i^{**} = \bigwedge_{j=1}^{k} m_j^{**}$ where $m_j^{**} \neq m_l^{**}$ for $j \neq l$; $n_i^{**} = m_j^{**}$ for $i \in I_j \subset \{0,1,\ldots,n\}$, $j = 1,2,\ldots,k$; $I_j \cap I_l = \emptyset$ for $j \neq l$; $\bigcup_{j=1}^{k} I_j = \{0,1,\ldots,n\}$ and $|[m_j^{**}]\varphi \cap N| \geq |I_j|, j = 1, 2,\ldots,k$. Thus, we can write $\bigwedge_{j=1}^{k} m_j^{**} = \bigwedge_{j=1}^{k}(\bigwedge\{s^{**} : s \in [m_j^{**}]\varphi \cap N\}) = \bigwedge_{j=1}^{k}(\bigwedge\{s : s \in [m_j^{**}]\varphi \cap N\})^{**} = (\bigwedge_{j=1}^{k}(\bigwedge\{s : s \in [m_j^{**}]\varphi \cap N\}))^{**}$. From $|[m_j^{**}]\varphi \cap N| \geq |I_j|$ $(j = 1,2,\ldots,k)$ and $\bigcup_{j=1}^{k} I_j = \{0,1,\ldots,n\}$, it follows that $\bigwedge_{j=1}^{k}(\bigwedge\{s : s \in [m_j^{**}]\varphi \cap N\})$ is meet of at least $(n+1)$ different nonclosed elements. Hence, $\bigwedge_{j=1}^{k}(\bigwedge\{s : s \in [m_j^{**}]\varphi \cap N\}) = 0$. Thus we obtain $\bigwedge_{i=0}^{n} a_i = \sum\{\bigwedge_{i=0}^{n} n_i^{**} : n_i^{**} \in N_{\alpha_i}^{**}\} = 0$ which implies $\bigvee_{i=0}^{n} \alpha_i^* \supseteq \theta(0,1) = \nabla$ and $Con(S) \in \mathcal{B}_n$.

Ad (3): Suppose that $[1]\alpha_i \subseteq B(S)$ for $i \in I$ and $[1]\alpha_j \cap N \neq \emptyset$ for $j \in J$ where $I \neq \emptyset \neq J$, $I \cap J \neq \emptyset$ and $I \cup J = \{0,1,\ldots,n\}$. Using the previous part of the proof we obtain that $\bigvee_{i \in I} \alpha_i^* \supseteq \theta(\bigwedge_{i \in I} a_i, 1)$, where $a_i = \sum N_{\alpha_i \wedge \varphi}^{**}, i \in I$. Let $m_j \in [1]\alpha_j \cap N$ for $j \in J$. Then $\theta(m_j,1) \wedge \alpha_i = \Delta$ and $\alpha_i^* \supseteq \theta(m_j,1)$ for arbitrary $i \in I$ and $j \in J$. It follows that $\bigvee_{i \in I} \alpha_i^* \supseteq \theta(\bigwedge_{i \in I} a_i, 1) \vee \bigvee_{j \in J} \theta(m_j,1) = \theta(\bigwedge_{i \in I} a_i, 1) \vee \theta(\bigwedge_{j \in J} m_j, 1) = \theta(\bigwedge_{i \in I} a_i \wedge \bigwedge_{j \in J} m_j, 1)$. Next we will prove that $\bigwedge_{i \in I} a_i \wedge \bigwedge_{j \in J} m_j^{**} = 0$. Since $\bigwedge_{i \in I} a_i = \sum\{\bigwedge_{i \in I} n_i^{**} : n_i^{**} \in N_{\alpha_i}^{**}\}$, we can write $\bigwedge_{i \in I} a_i \wedge \bigwedge_{j \in J} m_j^{**} = \sum\{\bigwedge_{i \in I} n_i^{**} \wedge \bigwedge_{j \in J} m_j^{**} : n_i^{**} \in N_{\alpha_i}^{**}\}$. Since $m_j < m_j^{**}$ and $m_j \in [1]\alpha_j$, obviously $m_j^{**} \in N_{\alpha_j}^{**}$ $(j \in J)$. Repeating the same consideration as in the part (2) of this proof we obtain that $\bigwedge_{i \in I} n_i^{**} \wedge \bigwedge_{j \in J} m_j^{**} = 0$ for arbitrary $|I|$-tuple $(n_i^{**} : n_i^{**} \in N_{\alpha_i}^{**} i \in I)$. Therefore, $\bigwedge_{i \in I} a_i \wedge \bigwedge_{j \in J} m_j \leq \bigwedge_{i \in I} a_i \wedge \bigwedge_{j \in J} m_j^{**} = 0$ and $\bigvee_{i=0}^{n} \alpha_i^* \supseteq \bigvee_{i \in I} \alpha_i^* \supseteq \theta(0,1) = \nabla$, hence $Con(S) \in \mathcal{B}_n$. ■

Corollary 3.13. *Let S be a* PCS *such that $B(S)$ is a complete Boolean algebra. For arbitrary $n \geq 1$ the following statesments are equivalent:*

(i) *$Con(S)$ is an (L_n)-lattice,*

(ii) *S is an (S_n)-semilattice.*

4. PSEUDOCOMPLEMENTED SEMILATTICES WITH RELATIVE (L_n)-CONGRUENCE LATTICES

Definition 4.1 ([2], Definition 2). Let L be a distributive lattice. L is said to be a *relative (L_n)-lattice* if every interval $[a,b]$ in L is an (L_n)-lattice.

Lemma 4.2 ([2], Theorem 2). *Let L be a distributive lattice with* 1. *The following conditions are equivalent:*

(i) *L is a relative (L_n)-lattice,*

(ii) *for every $a \in L$, $[a, 1]$ is an (L_n)-lattice.*

Lemma 4.3. *Let S be a* PCS. *Then S is an (S_n)-semilattice ($n \geq 1$) iff the quotient semilattice S/θ is an (S_n)-semilattice for arbitrary $\theta \in Con(S)$.*

Proof. Let S be a PCS. Suppose that S is an (S_n)-semilattice for some $n \geq 1$. We claim that for arbitrary $\theta \in Con(S)$ the following is true: if $[a]\theta \in N(S/\theta)$ then $[a]\theta \subseteq N(S)$.

Suppose that $[a]\theta \neq ([a]\theta)^{**} = [a^{**}]\theta$ and there exists $x \in [a]\theta$ such that $x = x^{**}$. Then $[a]\theta = [x]\theta = [x^{**}]\theta = ([x]\theta)^{**} = ([a]\theta)^{**}$ which is a contradiction to our assumption.

Let $[x_1]\theta, [x_2]\theta, \dots, [x_{n+1}]\theta \in S/\theta$ be such that $[x_i]\theta \neq [x_i^{**}]\theta$ $i = 1, \dots, n+1$ and $[x_i]\theta \neq [x_j]\theta$, $i \neq j$. From the previous part of proof follows that x_i $(i = 1, \dots, n+1)$ are pairwise distinct non-closed elements from S. Since S is an (S_n)-semilattice we obtain $\bigwedge_{i=1}^{n+1} [x_i]\theta = \left[\bigwedge_{i=1}^{n+1} x_i\right]\theta = [0]\theta$. Thus S/θ satisfies the condition (C_n).

Since $Con(S/\theta) \cong [\theta, \nabla] \subseteq Con(S)$ the congruence distributivity of S implies that the condition (D) is satisfied also in the quotient semilattice S/θ. The sufficient condition is obvious. ■

From the previous result and from Theorem 3.28 of [5], we immediately obtain

Corollary 4.4. *Let S be a* PCS. *The following statesments are equivalent*:

(i) *$Con(S)$ is a relative Stone lattice,*

(ii) *S satisfies (C_1) and for arbitrary congruence $\theta \in Con(S)$ the quotient* PCS *S/θ satisfies*:

$(S_{2(\theta)})$
 (a) if $A \subseteq N^{**}(S/\theta)$, then $\sum A$ exists;
 (b) if $K \subseteq C(S/\theta)$, then $\sum K$ exists. ■

Corollary 4.5. *Let S be a* PCS *such that the Boolean algebra $B(S/\theta)$ is complete for arbitrary congruence $\theta \in Con(S)$. For arbitrary $n \geq 1$ the following statesments are equivalent:*

(i) *$Con(S)$ is an (L_n)-lattice,*

(ii) $Con(S)$ *is a relative* (L_n)*-lattice,*

(iii) S *is an* (S_n)*-semilattice.* ∎

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Received 27 October 1998
Revised 1 October 1999