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RELATIVELY COMPLEMENTED ORDERED SETS

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Abstract

We investigate conditions for the existence of relative complements in ordered sets. For relatively complemented ordered sets with 0 we show that each element $b \neq 0$ is the least one of the set of all upper bounds of all atoms contained in b.

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Let $(A, \leq)$ be an ordered set and $B \subseteq A$. Denote by

$$L(B) = \{x \in A;\ x \leq b \text{ for all } b \in B\},$$
$$U(B) = \{x \in A;\ b \leq x \text{ for all } b \in B\}.$$

If $B = \{b_1, \ldots, b_n\}$, we shall write briefly $L(b_1, \ldots, b_n)$ or $U(b_1, \ldots, b_n)$ instead of $L(B)$ or $U(B)$, respectively. Moreover, for $B, C \subseteq A$ we write $L(B, C)$ for $L(B \cup C)$ and $U(B, C)$ for $U(B \cup C)$. Following [5], an ordered set $(A, \leq)$ is *modular* if for every $a, b, c \in A$ it holds:

$$a \leq c \Rightarrow L(c, U(a, b)) = L(U(a, L(b, c)))\,.$$

Modular ordered sets were treated in [2], a special sort of them, the so called distributive ordered sets were investigated in [2] and [4].

Let $(A, \leq)$ be an ordered set and $a \in A$. An element $b \in A$ is called a *complement of a* if

$$L(U(a, b)) = A \quad \text{and} \quad U(L(a, b)) = A\,.$$

Complemented ordered sets were studied in [1]. A generalization of the complement called a pseudocomplement in an ordered set was introduced in [5].

It is well known that if L is a complemented modular lattice, then L is also relatively complemented. The aim of our paper is to find a generalization of this result for ordered sets. However, there are several possibilities how to introduce the concept of a relative complement in an ordered set. We can pick up the following two:

Definition. Let $(A, \leq)$ be an ordered set, $a, b \in A$ and $a \leq b$. Let $x \in [a, b] = \{z \in A : a \leq z \leq b\}$. An element $y \in [a, b]$ is called a *weak relative complement of x in $[a, b]$* if

$$U(x, y) \cap [a, b] = \{b\} \qquad \text{and}$$
$$L(x, y) \cap [a, b] = \{a\}.$$

An element $y \in [a, b]$ is called a *strong relative complement of x in $[a, b]$* if

$$U(x, y) = U(b) \quad \text{and} \quad L(x, y) = L(a).$$

An ordered set $(S, \leq)$ is *strongly relatively complemented* if for every interval $[a, b]$ of S, each $x \in [a, b]$ has a strong relative complement in $[a, b]$.

Of course, every strong relative complemnt of $x \in [a, b]$ is also a weak relative complement of x in $[a, b]$ but not vice versa.

For the sake of brevity, we will write $U_{[a,b]}(x, y)$ or $L_{[a,b]}(x, y)$ instead of $U(x, y) \cap [a, b]$ or $L(x, y) \cap [a, b]$, respectively.

Theorem 1. *Let $(S, \leq)$ be a modular ordered set. Let $a, b \in S$, $a \leq b$, and $x \in [a, b]$. Suppose $y \in S$ is a complement of x. The set $U(a, L(y, b))$ has the least element p if and only if the set $L(U(a, y), b)$ has the greatest element p; in such a case, p is a strong relative complement of x in $[a, b]$.*

Proof. Denote by $A = U(a, L(y, b))$ and $B = L(U(a, y), b)$. Since $(S, \leq)$ is modular, we have

$$A = U(a, L(y, b)) = U(L(U(a, y), b)) = U(B),$$
$$B = L(U(a, y), b) = L(U(a, L(y, b))) = L(A).$$

(Let us note that the second line follows by an application of the dual of modular law since modularity is selfdual, see [2], [3].) Hence, if p is the least element of A, then $A = U(p)$ and $B = L(A) = L(U(p)) = L(p)$, thus p is the greatest element of B. Dually we can show the converse implication. Moreover, the modularity of $(S, \leq)$ yields

$$\begin{aligned} U(x,p) &= U(x, L(p)) = U(x, B) = U(x) \cap U(B) = U(x) \cap A = \\ &= U(x) \cap U(a, L(y,b)) = U(x, a, L(y,b)) = U(x, L(y,b)) = \\ &= U(L(U(x,y), b)) = U(L(b)) = U(b)\,, \\ L(x,p) &= L(x, U(p)) = L(x, A) = L(x) \cap L(A) = L(x) \cap B = \\ &= L(x) \cap L(U(a,y), b) = L(x, U(a,y), b) = L(x, U(a,y)) = \\ &= L(U(a, L(x,y))) = L(U(a)) = L(a)\,. \end{aligned}$$

■

Example 1. Applying methods of [2], we can check that the set $(S, \leq)$ in Figure 1 is modular.

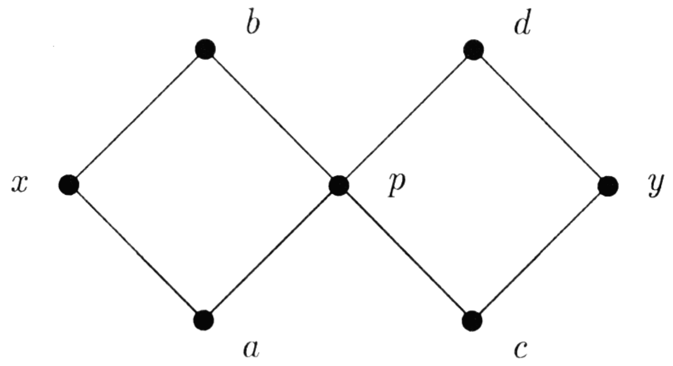

Figure 1

Of course, $L(U(x,y)) = L(\emptyset) = S$ and $U(L(x,y)) = U(\emptyset) = S$; thus y is a complement of x in $(S, \leq)$. Further, $U(a, L(y,b)) = U(a,c) = U(p)$. Thus $(S, \leq)$, a, b, x, y satisfy the assumption of Theorem 1, and hence p is a strong relative complement of x in $[a, b]$.

Example 2. Let $(S, \leq)$ be the ordered set depicted in Figure 2. S is modular and the element y is a complement of x. The set $A = U(a, L(y,b)) = U(a,c) = \{b, d\}$ has not a least element. The set $B = L(U(a,y), b) = L(d,b) = \{a, c\}$ has not a greatest element. It is easy to see that the element x has not a weak relative complement in $[a, b]$.

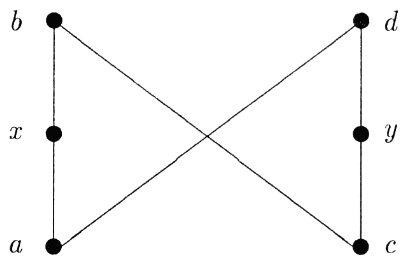

Figure 2

Although Theorem 1 is a generalization of the well known lattice statement, we can remove the assumption of modularity of $(S,\leq)$ and the complementarity of x to obtain a bit more general result:

Theorem 2. *Let $(S,\leq)$ be an ordered set, let $a,b\in S$, $a\leq b$, and $x\in[a,b]$. If there exists an element $y\in S$ such that:*

(i) *the set $L(U(a,y),b)$ has the greatest element e and the set $U(a,L(y,b))$ has the least element f,*

(ii) *the set $L(U(a,y),x)$ has the greatest element a and the set $U(x,L(y,b))$ has the least element b,*

then e and f are strong relative complements of x in $[a,b]$.

Proof. Set $A=U(a,L(y,b))$ and $B=L(U(a,y),b)$. By (i), there exist $e,f\in S$ with $A=U(f)$, $B=L(e)$. Prove $f\leq e$:

Since $c\leq b$ for each $c\in L(y,b)$, we have $U(b)\subseteq U(L(y,b))$. However, $a\leq b$ yields $U(b)\subseteq U(a)$, thus $U(b)\subseteq L(a,L(y,b))$, whence

$$(*)\qquad L(b)\supseteq L(U(a,L(y,b)))\,.$$

Analogously, $c\leq y$ for each $c\in L(y,b)$ yields $U(y)\subseteq U(L(y,b))$, clearly $U(a,y)\subseteq U(a,L(y,b))$, whence

$$(**)\qquad L(U(a,y))\supseteq L(U(a,L(y,b)))\,.$$

Applying $(*)$ and $(**)$, we conclude

$$L(e)=B=L(U(a,y),b)\supseteq L(U(a,L(y,b)))=L(A)=L(U(f))=L(f)\,,$$

i.e. $L(e)\supseteq L(f)$ proving $f\leq e$.

Moreover, (i) and (ii) imply

$$\begin{aligned} L(e,x) &= L(e)\cap L(x) = B\cap L(x) = L(U(a,y),b)\cap L(x) = \\ &= L(U(a,y),b,x) = L(U(a,y),x) = L(a)\,, \\ U(f,x) &= U(f)\cap U(x) = A\cap U(x) = U(a,L(y,b))\cap U(x) = \\ &= U(a,L(y,b),x) = U(L(y,b),x) = U(b)\,. \end{aligned}$$

Further, we obtain

$$\begin{aligned} U(b) = U(b,L(U(a,y),b)) = U(b,B) = U(b,e) \subseteq U(x,e) \subseteq U(x,f) = U(b), \\ L(a) = L(a,U(a,L(y,b))) = L(a,A) = L(a,f) \subseteq L(x,f) \subseteq L(x,e) = L(a), \end{aligned}$$

proving $U(x,e) = U(b)$ and $L(x,f) = L(a)$. Thus e and f are strong relative complements of x in $[a,b]$. ∎

Example 3. It is easy to see that the ordered set $(S,\leq)$ in Figure 3 is not modular and for x,y,a,b we have $x\in[a,b]$ and

$$\begin{aligned} L(U(a,y),b) &= L(e), \\ U(a,L(y,b)) &= U(f), \\ L(U(a,y),x) &= L(a), \\ U(x,L(y,b)) &= U(b)\,. \end{aligned}$$

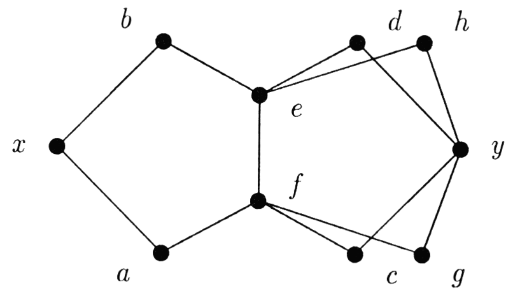

Figure 3

Hence, by Theorem 2, e and f are strong relative complements of x in $[a,b]$.

An anologous result is valid also for weak relative complements:

Theorem 3. *Let $(S,\leq)$ be an ordered set, let $a,b\in S$, $a\leq b$, and $x\in[a,b]$. If there exists an element $y\in S$ such that:*

(i) *the set* $L(U(a,y),b)$ *has the greatest element* e *and the set* $U(a,L(y,b))$ *has the least element* f,

(ii)* *the set* $L(U(a,y),x)$ *has a maximal element* a *and the set* $U(x,L(y,b))$ *has a minimal element* b,

then e *and* f *are weak relative complements of* x *in* $[a,b]$.

Proof. The proof of $f \leq e$ is the same as in that of Theorem 2.
Applying (i) and (ii)* we obtain

$$\begin{aligned} L_{[a,b]}(e,x) &= L(e) \cap L(x) \cap [a,b] = B \cap L(x) \cap [a,b] = \\ &= L(U(a,y),b) \cap L(x) \cap [a,b] = L(U(a,y),b,x) \cap [a,b] = \\ &= L(U(a,y),x) \cap [a,b] = L_{[a,b]}(U(a,y),x) = L_{[a,b]}(a) = \{a\} \end{aligned}$$

and dually

$$\begin{aligned} U_{[a,b]}(f,x) &= U(f) \cap U(x) \cap [a,b] = A \cap U(x) \cap [a,b] = \\ &= U(a,L(y,b)) \cap U(x) \cap [a,b] = U(a,L(y,b),x) \cap [a,b] = \\ &= U(L(y,b),x) \cap [a,b] = U_{[a,b]}(L(y,b),x) = U_{[a,b]}(b) = \{b\}\,. \end{aligned}$$

Since $f \leq e$, we conclude

$$\begin{aligned} &U_{[a,b]}(b) = U(b) \cap [a,b] = U(b,L(U(a,y),b)) \cap [a,b] = U(b,B) \cap [a,b] = \\ &= U(b,e) \cap [a,b] \subseteq U(x,e) \cap [a,b] \subseteq U(x,f) \cap [a,b] = U(b) \cap [a,b] = U_{[a,b]}(b), \end{aligned}$$

whence $U_{[a,b]}(x,e) = U_{[a,b]}(b)$. Dually, it can be shown that $L_{[a,b]}(x,f) = L_{[a,b]}(a)$. We have proved that e and f are weak relative complements of x in $[a,b]$. ■

Example 4. Consider the ordered set $(S,\leq)$ depicted in Figure 4. Although $(S,\leq)$ is not modular, the elements a,b,x,y satisfy (i) and (ii)* of Theorem 3, thus e and f are weak relative complements of x in $[a,b]$. Moreover, the element x has no strong relative complement in $[a,b]$, since $[a,b] = \{a,b,,x,e,f\}$ and for the only possible candidates e and f we have

$$\begin{aligned} L(e,x) &= \{a,y\} \neq \{a\} = L(a), \\ U(e,x) &= \{b,h\} \neq \{b\} = U(b)\,, \end{aligned}$$

analogously also for f.

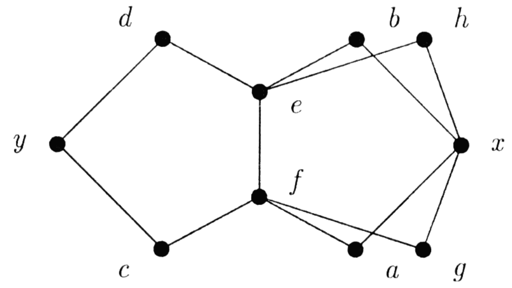

Figure 4

Now, we turn our attention to some aspects of atomicity in relatively complemented ordered sets. In the case of lattices, it is well known that if L is a relatively complemented lattice of finite length and $b \in L$, $b \neq 0$, then $b = \vee A(b)$, where $A(b)$ is the set of all atoms of L less or equal to b (see e.g. [6]). In the case of ordered sets with 0 we investigate whether $U(A(b)) = U(b)$.

An element a of an ordered set $(S, \leq)$ is called an *atom* if either a covers 0 whenever 0 is the least element of $(S, \leq)$ or a is a minimal element of $(S, \leq)$ in the opposite case. For $b \in S$, denote by $A(b)$ the set of all atoms of S below b. $(S, \leq)$ is called *atomic* if for any $b \in S$, $b \neq 0$ (whenever 0 in S exists) there exists an atom $a \in S$ with $a \leq b$. It is almost evident that if $(S, \leq)$ is of a finite length, then $(S, \leq)$ is atomic.

Theorem 4. *Let $(S, \leq)$ be a strongly relatively complemented ordered set of a finite length with 0. If $b \in S$ and $b \neq 0$, then $U(A(b)) = U(b)$.*

Proof. If b is an atom in S, then $A(b) = \{b\}$, and hence $U(A(b)) = U(b)$. Suppose b is not an atom in S and $b \neq 0$. Since $(S, \leq)$ is of a finite length and hence atomic, there exists $p_1 \in A(b)$. Since $(S, \leq)$ is strongly relatively complemented, there exists $c_1 \in [0, b]$ with $U(p_1, c_1) = U(b)$.

(a) If c_1 is an atom of $(S, \leq)$ then $c_1 \leq b$ implies $c_1 \in A(b)$. Denote by $D = A(b) \setminus \{p_1, c_1\}$. Clearly $U(D) \supseteq U(b)$ (since $U(D) = S$ if $D = \emptyset$ and, for $D \neq \emptyset$, $d \leq b$ for each $d \in D$. Then

$$U(A(b)) = U(p_1, c_1) \cap U(D) = U(b) \cap U(D) = U(b).$$

(b) Suppose c_1 is not an atom of $(S, \leq)$. We can repeate the same consideration for the element c_1 (instead of the element b), i.e. there exists

$p_2 \in A(c_1)$ and $c_2 \in [0, c_1]$ such that $U(p_2, c_2) = U(c_1)$. Since $(S, \leq)$ is of a finite length, we will finish after n steps of this procedure to obtain an element $c_n \in S$ such that $U(p_n, c_n) = U(c_{n-1})$ and $c_n \in A(c_{n-1})$. Denote by $D_n = A(c_{n-1}) \setminus \{p_n, c_n\}$. Evidently, $U(D_n) \supseteq U(c_{n-1})$ and

$$U(A(c_{n-1})) = U(p_n, c_n) \cap U(D_n) = U(c_{n-1}) \cap U(D_n) = U(c_{n-1})\,.$$

Further, let $D_{n-1} = A(c_{n-2}) \setminus \{p_{n-1}, A(c_{n-1})\}$. Clearly $U(D_{n-1}) \supseteq U(c_{n-2})$ and again

$$\begin{aligned} U(A(c_{n-2})) &= U(p_{n-1}) \cap U(A(c_{n-1})) \cap U(D_{n-1}) = \\ &= U(p_{n-1}, c_{n-1}) \cap U(D_{n-1}) = U(c_{n-2}) \cap U(D_{n-1}) = U(c_{n-2})\,. \end{aligned}$$

Analogously we proceed to prove $U(A(c_k)) = U(c_k)$ for $k = 1, \dots, n$. For $D_1 = A(b) \setminus \{p_1, A(c_1)\}$ we have $U(D_1) \supseteq U(b)$, thus also

$$\begin{aligned} U(A(b)) &= U(p_1) \cap U(A(c_1)) \cap U(D_1) = \\ &= U(p_1) \cap U(c_1) \cap U(D_1) = U(b) \cap U(D_1) = U(b)\,. \end{aligned}$$

■

Example 5. Let $(S, \leq)$ be the ordered set depicted in Figure 5. Then $(S, \leq)$ is strongly relatively complemented and of a finite length. For the element $b \in S$ we really have $A(b) = \{a, c, d\}$ and $U(A(b)) = U(a, c, d) = U(b)$.

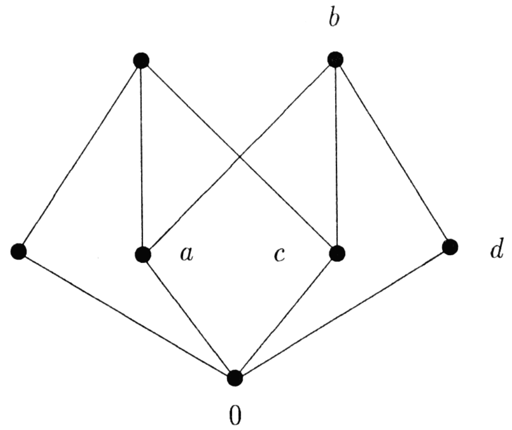

Figure 5

Remark. If $(S, \leq)$ is a strongly relatively complemented ordered set of a finite length without 0, the assertion of Theorem 4 does not hold in general, see, e.g., Figure 6, where $A(b) = \{a, c\}$ but $U(A(b)) = U(a, c) = U(d) = \{d, b\} \neq \{b\} = U(b)$.

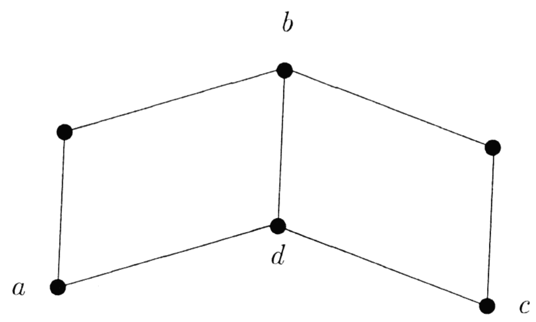

Figure 6

Moreover, b is not even a minimal element of $U(A(b))$. On the other hand, we can state the following

Theorem 5. *Let $(S, \leq)$ be an ordered set, let $b \in S$ and $b \neq 0$ whenever 0 in S exists. If b covers an atom a and* card$(A(b)) \geq 2$, *then b is a minimal element of $U(A(b))$.*

Proof. Let $b \neq 0$ covers an atom a and let *card*$(A(b)) \geq 2$. Suppose b is not minimal in $U(A(b))$. Then there exists $m \in U(A(b))$ with $m < b$. Since $a \in A(b)$, we conclude $a \leq m < b$. But b covers a, i.e. $a = m$, hence $a \in U(A(b))$ Since *card*$(A(b)) \geq 2$, there exists $c \in A(b)$ with $c \neq a$. It is easy to check that c and a are bot comparable. However $a \in U(A(b))$ implies $c \leq a$, a contradiction. ■

Example 6. Let $(S, \leq)$ be an ordered set with the diagram depicted in Figure 7.

Then $(S, \leq)$ has not 0 and it is not of a finite length. $A(b) = \{a, c\}$, i.e. *card*$(A(b)) = 2$. In accordance with Theorem 5, b is a minimal element in the set $U(A(b)) = U(a.c) = \{b, d_1, d_2, \ldots, e_1, e_2, \ldots\}$. On the other hand $U(A(b)) \neq \{b\} = U(b)$.

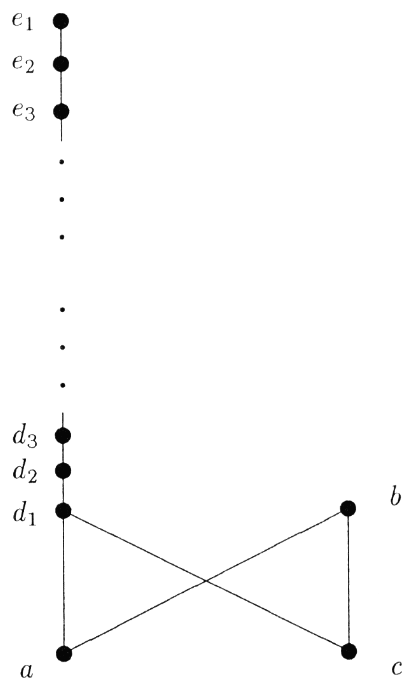

Figure 7

Remark. The condition "b covers an atom a" in Theorem 5 is not necessary. For the set $(S, \leq)$ in Figure 8 we have that b is the unique and hence minimal element of $U(A(b))$ but b covers no atom of S.

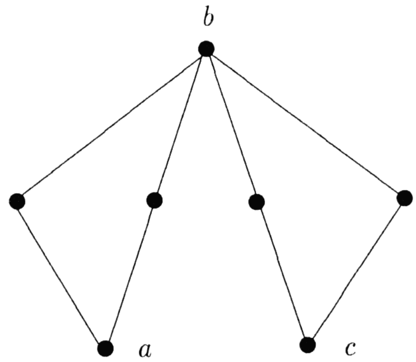

Figure 8

On the other hand, if $b \neq 0$ (whenever 0 in S exists) and b is not an atom of S, then, if b is minimal in $U(A(b))$, $card\,(A(b)) \geq 2$. Namely, if $card\,(A(b)) = 0$, then $A(b) = \emptyset$ and $U(A(b)) = U(\emptyset) = S$; thus b is a minimal element of S. Since $b \neq 0$, S has not 0. However, b is not an atom, a contradiction. If $card\,(A(b)) = 1$, then $A(b) = \{a\}$ and $a \neq b$, i.e. $U(A(b)) = U(a)$ and b cannot be the minimal element of $U(A(b))$, a contradiction again.

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