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A FACTORIZATION OF ELEMENTS IN $PSL(2,F)$, WHERE $F = \mathbb{Q}, \mathbb{R}$

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Abstract

Let G be a group and $K_n = \{g \in G : o(g) = n\}$. It is prowed: (i) if $F = \mathbb{R}$, $n \geq 4$, then $PSL(2,F) = K_n^2$; (ii) if $F = \mathbb{Q}, \mathbb{R}$, $n = \infty$, then $PSL(2,F) = K_n^2$; (iii) if $F = \mathbb{R}$, then $PSL(2,F) = K_3^3$; (iv) if $F = \mathbb{Q}, \mathbb{R}$, then $PSL(2,F) = K_2^3 \cup E, E \notin K_2^3$, where E denotes the unit matrix; (v) if $F = \mathbb{Q}$, then $PSL(2,F) \neq K_3^3$.

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Let G be a group and $K_n = K_n(G) = \{g \in G : o(g) = n\}$. Let $SL(m,F)$ and $PSL(m,F)$ be a *special linear* or *projective specjal linear* (resp.) groups of degree m over a field F. Many papers have been devoted to the powers of the set K_2 (see [3] – [9]) but only few papers have been written about the powers of the set K_n for $n > 2$ (see [1] – [3]). In the papers [3] and [5], it has been proved that if F is an algebraically closed field, then $PSL(3,F) = K_nK_n$ for $n > 2$ and $PSL(3,F) = K_2^4$ for any F. Note that we do not identify K_2 with the set of involutions. In the paper [7], it has been proved that if $F = \mathbb{Q}$, $\mathbb{R}$, where $\mathbb{Q}$ denotes the field of rational numbers and $\mathbb{R}$ denotes the field of real numbers, then $PSL(2,F) = K_n^4$.

In this paper we will prove the following properties:

(i) if $F = \mathbb{R}$, $n \geq 4$, then $PSL(2,F) = K_n^2$;

(ii) if $F = \mathbb{Q}, \mathbb{R}$, $n = \infty$, then $PSL(2,F) = K_n^2$;

(iii) if $F = \mathbb{R}$, then $PSL(2,F) = K_3^3$;

(iv) if $F = \mathbb{Q}$ or $\mathbb{R}$, then $PSL(2, F) = K_2^3 \cup E, E \notin K_2^3$, where E denotes the unit matrix;

(v) if $F = \mathbb{Q}$, then $PSL(2, F) \neq K_3^3$.

Recall, that $PSL(2, \mathbb{C}) = K_n^2$, where $\mathbb{C}$ denotes the field of complex numbers (see [2]).

We begin with some lemmas.

Lemma 1. *Let F be any field. In $SL(2, F)$, each non-scalar matrix is similar to a matrix of the form* $\begin{bmatrix} 0 & r \\ -r^{-1} & s \end{bmatrix} = D$. *The order of D depends only on s.*

If $F = \mathbb{R}$, then

a) *the order of the matrix $D \in SL(2, \mathbb{R})$ is $n > 2$ iff $s = 2\cos\frac{2k\pi}{n}$ and $(k, n) = 1$;*

b) *the order of the matrix $D \in PSL(2, \mathbb{R})$ is $n > 2$ iff $s = 2\cos\frac{k\pi}{n}$ and $(k, n) = 1$ or $s = 2\cos\frac{2k\pi}{n}$, $(k, n) = 1$.*

If $F = \mathbb{Q}$ or $\mathbb{R}$ and $|s| > 2$, then the order of D is ∞.

Proof. If F is any field, then for each $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, F)$ there exists a matrix

$$X = \begin{bmatrix} x & y \\ \frac{1}{r}(xa + cy) & \frac{1}{r}(bx + yd) \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & r \\ -r^{-1} & s \end{bmatrix}$$

such that $A = X^{-1}DX$ and $s = a + d$. The condition $\det X = 1$ holds since the equation $\frac{x}{r}(bx + yd) - \frac{y}{r}(xa + cy) = 1$ has a solution in r, x, y.

If F is any field, then we can find that

$$D^n = \begin{bmatrix} \varphi_{n-2}(s) & r\psi_{n-1}(s) \\ -r^{-1}\psi_{n-1}(s) & \omega_n(s) \end{bmatrix},$$

where $\varphi_{n-2}, \psi_{n-1}, \omega_n$ are polynomials in s which means that the order of D depends only on s.

In the case $F = \mathbb{R}$, it is easy to notice that the order of any matrix A over $\mathbb{R}$ is the same as over $F = \mathbb{C}$. Thus if $-2 < s < 2$, we can put $s = 2\cos\varphi$, and then the matrix D is similar to the diagonal matrix $\begin{bmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{bmatrix}$ over $\mathbb{C}$. Hence, the rest of the proof follows from obvious properties of the group of the n-th roots of unity. If $|s| > 2$, then the order of D is ∞. ∎

Lemma 2 (*see* [5]). *If $V=\operatorname{diag}(v_1,...,v_m)$, $W=\operatorname{diag}(w_1,...,w_m)$, $v_i\neq v_j$, $w_i\neq w_j$ for $i\neq j$ and $V,W\in SL(m,F)$, then $SL(m,F)=C_VC_W\cup Z$, where C_V denotes the conjugacy class of V and Z denotes the center of $SL(m,F)$.* ■

Lemma 3. *If*

$$N_i=\begin{bmatrix}0 & w_i\\ -w_i^{-1} & 0\end{bmatrix},\quad T_i=\begin{bmatrix}0 & 1\\ -1 & x_i\end{bmatrix},\quad N_i,T_i\in SL(2,F),$$

then the trace $\operatorname{tr}(N_1^{T_1}N_2^{T_2}N_3^{T_3})=s$ is any arbitrary element of F, where $(N_i^{Ti}=T_i^{-1}N_iT_i)$.

Proof. If we put $x_1=x_2=0$, then $s=-w_3w_1^{-1}w_2^{-1}(w_1^2+w_2^2)x_3$. Thus s is directly proportional to x_3 and s can be any arbitrary element of F. ■

Lemma 4. *If*

$$M_i=\begin{bmatrix}0 & w_i\\ -w_i^{-1} & d_i\end{bmatrix}(i=1,2,3),\quad T_i=\begin{bmatrix}0 & 1\\ -1 & x_i\end{bmatrix},\quad d_i\neq 0,$$

and $M_i,T_i\in SL(2,F)$, then there are w_i such that the trace $\operatorname{tr}(M_1M_2^{T_2}M_3^{T_3})=s$ is any arbitrary element of F.

Proof. A calculation shows that if we take $w_2=-d_2d_1^{-1}w_1^{-1}, x_3=x_2+d_3w_3^{-1}$and $(w_1w_3d_1)^2\neq d_2^2$, then $s=x_2(d_1d_2^{-1}w_3-d_1d_2^{-1}w_1^{-1}w_3^{-1})+d_1d_3d_2^{-1}$, so s varies as a linear function of x_2. ■

Lemma 5. *Let $M_i=\begin{bmatrix}0 & w_i\\ -w_i^{-1} & d_i\end{bmatrix}$, $S_i=\begin{bmatrix}0 & y_i\\ -y_i^{-1} & x_i\end{bmatrix}$, over $\mathbb{R}$. Then*

$$\begin{aligned}s=\operatorname{tr}(M_1^{S_1}M_2^{S_2})&=-w_1w_2\left(\frac{x_1y_1-x_2y_2}{y_2y_1}\right)^2+\\ &(x_1y_1-x_2y_2)\left(\frac{d_1w_2}{y_2^2}-\frac{w_1d_2}{y_1^2}\right)-\left(\frac{w_2}{w_1}\right)\left(\frac{y_1}{y_2}\right)^2-\left(\frac{w_1}{w_2}\right)\left(\frac{y_2}{y_1}\right)+d_1d_2.\end{aligned}\tag{1}$$

achieves the minimum

$$s_{\min}=\frac{1}{2}\sqrt{(4-d_1^2)(4-d_2^2)}+\frac{1}{2}d_1d_2$$

and the maximum value

$$s_{\max} = -\frac{1}{2}\sqrt{(4-d_1^2)(4-d_2^2)} + \frac{1}{2}d_1d_2$$

for $w_1w_2 < 0$ *and* $w_1w_2 > 0$, *respectively.*

Proof. If we consider the trace s as a function of x_1, x_2, then the condition

$$\frac{\partial s}{\partial x_1} = \frac{\partial s}{\partial x_2} = 0 \tag{2}$$

is equivalent to the condition

$$2(x_1y_1 - x_2y_2) = \frac{d_1}{d_2}y_1^2 - \frac{d_2}{d_1}y_2^2. \tag{3}$$

Since $\frac{\partial^2 s}{\partial x_1^2} = -\frac{2w_1w_2}{y_2^2}$, $\frac{\partial^2 s}{\partial x_2^2} = -\frac{2w_1w_2}{y_1^2}$, $\frac{\partial^2 s}{\partial x_1 \partial x_2} = -\frac{2w_1w_2}{y_1y_2}$, therefore

$$s(x_1+h, x_2+k) - s(x_1, x_2) = -\frac{2w_1w_2}{y_1^2y_2^2}(y_1 - y_2k)^2. \tag{4}$$

Hence, $s(x_1, x_2)$ achieves the minimum and the maximum value for $w_1w_2 < 0$ and $w_1w_2 > 0$, respectively. The value of the trace s , at the surface (3) equals

$$\frac{1}{4}x(d_2^2 - 4) + \frac{1}{4}(d_1^2 - 4)\frac{1}{x} + \frac{1}{2}d_1d_2, \tag{5}$$

where $x = \frac{w_1}{w_2}(\frac{y_1}{y_2})^2$.

The function (5) in x and, as a result, also s achieves the minimum $s_{\min}$ and the maximum $s_{\max}$ value for

$$\frac{w_1}{w_2} = -(\frac{y_1}{y_2})^2\sqrt{\frac{d_1^2-4}{d_2^2-4}} \quad \text{and} \quad \frac{w_1}{w_2} = (\frac{y_1}{y_2})^2\sqrt{\frac{d_1^2-4}{d_2^2-4}},$$

respectively. ■

Lemma 6. *If* $F = \mathbb{R}$, *then the non-scalar matrix* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, F)$ *and* $D = \begin{bmatrix} 0 & r \\ -r^{-1} & s \end{bmatrix}$, *are similar in* $SL(2, F)$ *provided* $s = a + d$, $br \geq 0$ *or* $-cr \geq 0$.

Proof. We have $XAX^{-1} = D$, where

$$X = \begin{bmatrix} x & y \\ \frac{1}{r}(ax+cy) & \frac{1}{r}(bx+yd) \end{bmatrix}, \quad \det X \neq 0.$$

The condition $X \in SL(2, F)$ is equivalent to the solvability of the equation

$$bx^2 + xy(d-a) - cy^2 - r = 0 \text{ in } x \text{ or } y. \tag{6}$$

The descriminant $\Delta = y^2(s^2-4)+4br$ or $\Delta = x^2(s^2-4)-4cr$, respectively, must be a non negative element of F.

By the assumption $br \geq 0$ or $-cr \geq 0$, we can chose so small y or x such that $\Delta \geq 0$ for any $a, d \in \mathbb{R}$. ∎

Lemma 7. *Let* $s = \operatorname{tr}(M_1^{S_1} M_2^{S_2})$ *be defined by* (1) *and let* n *be the order of* M_i*. Then:*

if $n = 2$, *then* $-\infty < s \leq -2$ *or* $2 \leq s < \infty$;
if $n = 3$, *then* $-\infty < s \leq -1$ *or* $1 \leq s < \infty$;
if $n \geq 4$, *then* $-\infty < s < \infty$.

Proof. For $d_1 = 2\cos\frac{\pi}{n}$ and $d_2 = 2\cos\frac{\pi(n-1)}{n}$ the trace s achieves the minimum

$$s_{\min} = -2\cos\frac{2\pi}{n} \tag{7}$$

and for $d_1 = 2\cos\frac{\pi}{n}$ and $d_2 = 2\cos\frac{\pi}{n}$, the trace s achieves the maximum value

$$s_{\max} = 2\cos\frac{2\pi}{n}, \tag{8}$$

by Lemma 5. The rest of the proof follows from (7), (8) and definition (1) of s. ∎

Lemma 8 (see [4]). *Let* G *be a group. An element* $g \in K_2^m$ $(m \geq 2)$ *if and only if there is an element* $x \in K_2^{m-1}$, $x \neq g^{-1}$ *such that* $(gx)^2 = 1$. ∎

Theorem 1. $PSL(2, \mathbb{R}) = K_n^2$, *for* $n \geq 4$.

Proof. Let

$$M_i = \begin{bmatrix} 0 & w_i \\ -w_i^{-1} & d_i \end{bmatrix}, \quad T_1 = \begin{bmatrix} 0 & 1 \\ -1 & d_i \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and $d_i = 2\cos\frac{\pi j}{n}, i = 1, 2;\ (j, n) = 1$. From (1) for $x_2 = 0,\ y_1 = y_2 = 1$, it results that

$$s = -w_1 w_2 x_1^2 + (w_2 d_1 - w_1 d_2)x_1 - \frac{w_1}{w_2} - \frac{w_2}{w_1} + d_1 d_2. \tag{9}$$

The function (9) in x_1 achieve the same minimum and maximum value as the function (1). For this reason, the trace $\mathrm{tr}(M_1^{S_1} M_2^{S_2}) = s$ fulfills the condition of Lemma 7. The matrix

$$M_1^{T_1} M_2^{T_2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ where } b = -\frac{w_1}{w_2}x_1 + \frac{d_1}{w_2},\ \ c = w_1(-d_2 - w_2 x_1)$$

is similar in $GL(2, \mathbb{R})$ to the matrix

$$D = \begin{bmatrix} 0 & r \\ -r^{-1} & s \end{bmatrix},\ s = a + d \text{ for any } r \neq 0.$$

By Lemma 6, these matrices are similar in $SL(2, \mathbb{R})$ provided

$$rc \leq 0 \text{ or } rb \geq 0. \tag{10}$$

From Lemma 7, it results that the equation (9) is solvable in x_1 and

$$x_1^{'} = \frac{w_2 d_1 - w_1 d_2 + \sqrt{\triangle}}{2w_1 w_2},\quad x_1^{''} = \frac{w_2 d_1 - w_1 d_2 - \sqrt{\triangle}}{2w_1 w_2},$$

where $\triangle = (w_2 d_1 + w_1 d_2)^2 - 4w_1^2 - 4w_2^2 - 4w_1 w_2 s$.

If we put $x_1 = x_1^{'}$, then

$$b = \frac{1}{2w_2^2}\left(w_2 d_1 + w_1 d_2 - \sqrt{\triangle}\right) \text{ and } c = -\frac{1}{2}\left(w_2 d_1 + w_1 d_2 + \sqrt{\triangle}\right).$$

Note that $\triangle(-w_1, -w_2) = \triangle(w_1, w_2)$. Hence, if $r > 0$, then the signs of w_1 and w_2 can be chosen such that $w_2 d_1 + w_1 d_2 > 0$, thus $cr < 0$; if $r < 0$, then the signs of w_1 and w_2 can be chosen such that $w_2 d_1 + w_1 d_2 < 0$, thus $br > 0$. If $w_2 d_1 + w_1 d_2 = 0$, then $c < 0$ and $b < 0$, thus for $r > 0$, $rc < 0$ and for $r < 0$, $rb > 0$. Hence, condition (10) holds in all cases. Thus $M_1^{T_1} M_2^{T_2}$ and D are similar in $SL(2, \mathbb{R})$, by Lemma 6. Hence, matrices conjugate to D run over all non-scalar matrices of $PSL(2, \mathbb{R})$, by Lemma 1. Our set of matrices contains together with the matrix $L = \begin{bmatrix} 0 & r \\ -r^{-1} & d_i \end{bmatrix}$ also L^{-1}, so $E = LL^{-1} \in K_n^2$. Therefore, $K_n^2 = PSL(2, \mathbb{R})$. ■

Theorem 2. a) $PSL(2,\mathbb{R}) = K_3^3$,
b) $PSL(2,\mathbb{Q}) \neq K_3^3$,
c) $PSL(2,F) \neq K_2^3$ *for* $F = \mathbb{Q}$ *and* $\mathbb{R}$.

Proof. Let $M_i = \begin{bmatrix} 0 & z \\ -z^{-1} & d_i \end{bmatrix}$, where $d_i = 2\cos\frac{\pi j}{n}, i = 1,2;\ (j,n) = 1$ and M_i, T_i as in Lemma 3 or 4. If we take $r = x^2b + xy(d-a) - cy^2, x, y \in F$, then the matrices $M_1^{T_1}M_2^{T_2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $D = \begin{bmatrix} 0 & r \\ -r^{-1} & s \end{bmatrix}$, $s = a + d$ are similar in $SL(2,F)$.

Consider the matrix

$$M_iD = \begin{bmatrix} 0 & z \\ -z^{-1} & d_i \end{bmatrix} \cdot \begin{bmatrix} 0 & r \\ -r^{-1} & s \end{bmatrix} = \begin{bmatrix} -zr^{-1} & zs \\ -d_ir^{-1} & -rz^{-1} + d_is \end{bmatrix}.$$

By Lemma 3 or 4 the trace $\mathrm{tr}(M_iD) = t$ runs over all of F, according to $n = 2$ or $n = 3$. The matrix M_iD is similar in the gerneral linear group $GL(2,F)$ to the matrix $C=\begin{bmatrix} 0 & m \\ -m^{-1} & t \end{bmatrix}$. The similarity of M_iD and C in $SL(2,F)$ is equivalent to the condition

$$x^2(t^2-4) + 4\frac{md_i}{r} \geq 0, \tag{11}$$

by Lemma 6.

Since $d_i = \pm 1$ for $n = 3$, it is possible to chose d_i and x such that the condition (11) holds in $\mathbb{R}$. Hence, by the Lemma 1, matrices conjugate to C run over all non-scalar matrices of $PSL(2,F)$. By Lemma 7, K_3^2 contains the matrix $B=\begin{bmatrix} 0 & b \\ -b^{-1} & d_i \end{bmatrix} \in K_3$, where $\mathrm{d}_i = 2\cos\frac{\pi j}{3}, (j,3) = 1$. The set K_3 together with B contains also B^{-1}. Hence $E = BB^{-1} \in K_3^3$. Therefore $PSL(2,\mathbb{R}) = K_3^3$.

If $F = \mathbb{Q}$, then the condition (11) cannot hold for $t = 2$ and for any arbitrary $m \in \mathbb{Q}$. Hence $PSL(2,\mathbb{R}) \neq K_3^3$.

If $n = 2$, then $d_i = 0$ and the condition (11) cannot hold for $|t| < 2$ even for $F = \mathbb{R}$. Hence $PSL(2,F) \neq K_3^3$ for $F = \mathbb{Q}, \mathbb{R}$. The part b) of Theorem 2 follows.

The statement c) results from Lemma 8. Indeed, the set of non-scalar matrices of $K_2^2 \subset PSL(2,F)$ consist of matrices

$$X = \begin{bmatrix} 0 & x \\ -x^{-1} & 0 \end{bmatrix} \cdot \begin{bmatrix} y & z \\ -z^{-1}(1+y^2) & -y \end{bmatrix} \in K_2^2 \tag{12}$$

and their conjugates. The conditions $(XG)^2 = E,\ G = \pm E,\ X \neq G$ are equivalent to

$$x^2(y^2+1)+z^2 = 0; \quad x,y,z \neq 0, \tag{13}$$

which cannot be fulfilled over $\mathbb{Q}$ and $\mathbb{R}$. Hence $E \notin K_2^3$, by Lemma 8. ■

Theorem 3. *If $F = \mathbb{Q}$ or $\mathbb{R}$ and $n = \infty$, then $SL(2,F) = K_n^2$ and $PSL(2,F) = K_n^2$.*

Proof. Among matrices of order $n = \infty$ in $PSL(2,F)$ there are matrices of the form

$$A_i = \begin{bmatrix} 0 & a_i \\ -a_i^{-1} & d_i + d_i^{-1} \end{bmatrix},$$

with distinct eigenvalues d_i, d_i^{-1}, where $d_i \neq 0$. Observe that $o(A_i) = o(-A_i) = o(A_i^{-1}) = \infty$ and $K_n^2 = \bigcup_{i,j} C_{A_i}C_{A_j}$. Lemma 2 implies that $K_n^2 \cup Z = SL(2,F)$ but $E \in C_{A_i}C_{A_i^{-1}}$ and $-E \in C_{A_i}C_{(-A_i)}$, so $K_n^2 = SL(2,F)$.

The equality $K_\infty^2 = PSL(2,F)$ can be proved similarly. ■

From Theorems 1, 2, 3, all proporties (i) – (v) follow immediately.

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