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EQUIVALENT CONDITIONS FOR P-NILPOTENCE

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Abstract

In the first part of this paper we prove without using the transfer or characters the equivalence of some conditions, each of which would imply p-nilpotence of a finite group G. The implication of p-nilpotence also can be deduced without the transfer or characters if the group is p-constrained. For p-constrained groups we also prove an equivalent condition so that $\mathrm{O}^{q'}(G)P$ should be p-nilpotent. We show an example that this result is not true for some non-p-constrained groups.

In the second part of the paper we prove a generalization of a theorem of Itô with the help of the knowledge of the irreducible characters of the minimal non-nilpotent groups.

Keywords: p-nilpotent group, p-constrained group, character of a group, Schmidt group, Thompson-ordering, Sylow p-group.

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1 Definitions and known results

We know the remarkable theorems of Frobenius which tell that in Theorem 2.1 (i) and (iv) both imply that the finite group G has a normal p-complement. All existing proofs of them use the transfer homomorhism or characters.

The well-studied minimal non-nilpotent groups, i.e. non-nilpotent groups, each of whose subgroups are nilpotent, sometimes are called Schmidt groups or (p, q)-groups. They can be described without using the transfer, see 5.1 Satz and 5.2 Satz in pp. 280-281 of [6]. Let G be a minimal non-nilpotent group. Then it can be proved without using the transfer or characters, that

1. G is solvable.
2. $|G|$ is divisible only by 2 primes, say $|G| = p^a q^b$.
3. $G' = P \in \mathrm{Syl}_p(G)$.
4. If $p > 2$, then $\exp(P) = p$; if $p = 2$, then $\exp(P) \leq 4$; $\exp(\mathrm{Z}(P)) = p$ in all cases.
5. P is either abelian or $P' = \Phi(P) = \mathrm{Z}(P) \leq \mathrm{Z}(G)$.
6. If $Q \in \mathrm{Syl}_q(G)$, then Q is cyclic; and if $Q = \langle x \rangle$, then $\langle x^q \rangle \leq \mathrm{Z}(G)$.
7. If P is abelian, then Q acts irreducibly on P; if P is nonabelian, then Q acts irreducibly on $P/\mathrm{Z}(P)$. If P is abelian, then P is of exponent p; and if P is nonabelian, then $P/\mathrm{Z}(P)$ is also of exponent p. So, they can be considered as vector spaces over $GF(p)$. Their dimension is $\mathrm{o}(p)(\mathrm{mod}(q))$, which is even in the nonabelian case.
8. G is generated by its Sylow q-subgroups.

These groups are non-p-nilpotent. It can also be proved using the transfer or characters that a group is p-nilpotent if and only if, it does not contain such a subgroup.

We shall use the following:

Notation 11. We shall write $(p, q) \not\leq G$ if the group G does not contain a (p, q)-group, otherwise we write $(p, q) \leq G$.

Let us recall the definition of Thompson-ordering:

Definition 12. Let G be a finite group. Let $\mathcal{P}$ be a property of subgroups of G. Let $\mathcal{A} = \{A|$ p-subgroup of G, $\mathrm{N}_G(A) < G, A$ p-group, $\mathrm{N}_G(A)$ has the property $\mathcal{P}\}$. We tell for $A_1, A_2 \in \mathcal{A}$ that A_1 is smaller than A_2 in the Thompson-ordering if either $|\mathrm{N}_G(A_1)|_p < |\mathrm{N}_G(A_2)|_p$, or $|\mathrm{N}_G(A_1)|_p = |\mathrm{N}_G(A_2)|_p$ and $|A_1| < |A_2|$.

Definition 13. Let $P \in \mathrm{Syl}_p(G)$. A subgroup $U \leq P$ is called strongly closed if for every $u \in U$ if $u^x \in P$ then $u^x \in U$.

2 Main results

The aim of this paper is to prove that the equivalence of the following four conditions can be proved without the use of transfer or characters:

Theorem 21. *Let G be a finite group, let $P \in \mathrm{Syl}_p(G)$. Then the following are equivalent:*

(i) *If $x, y \in P$ are G-conjugate, then they are conjugate already in P;*

(ii) *If $x, y \in P$ of order p or 4 and are G-conjugate, then they are conjugate already in P;*

(iii) *$(p, q) \not\lesssim G$ for every prime $q \neq p$;*

(iv) *For every p-subgroup $U \leq G$, $\mathrm{N}_G(U)/\mathrm{C}_G(U)$ is a p-group.*

As a corollary we get:

Theorem 22. *Let G be a p-constrained group, $P \in \mathrm{Syl}_p(G)$. If any of the conditions of the above theorem holds for G, then one can deduce without using the transfer that G has a normal p-complement.*

As an application of Theorem 2.1 we prove also the following:

Theorem 23. *Let G be a finite group, let $p \neq q$ be primes with $p, q \in \pi(G)$. Let $P \in \mathrm{Syl}_p(G)$ and let $\mathrm{O}^{q'}(G)$ denote the subgroup of G generated by the q-elements of G. If G is p-constrained then the following are equivalent:*

(i) *$(p, q) \not\lesssim G$.*

(ii) *$\mathrm{O}^{q'}(G)P$ has a normal p-complement.*

Another application of Theorem 2.2 is to prove without using the transfer the following generalization of a theorem of Itô:

Theorem 24. *Let G be a finite p-constrained group, let $P \in \mathrm{Syl}_p(G)$. Let us suppose that χ is a character of G satisfying the following conditions:*

α) $\chi(1) \leq 2p - 2$,

β) *for every subgroup $H \leq G$, χ_H does not have a constituent of degree p,*

γ) $\mathrm{Ker}(\chi) = 1$.

Then one of the following two possibilities holds:

(i) *P is abelian and P is normal in G;*

(ii) *p is a Fermat-prime and one of the constituents of χ has degree at least $p - 1$.*

The inequality in α) is sharp. There is a solvable group G_0 having a character $\chi_0 \in \mathrm{Char}(G_0)$ satisfying β) and γ) with degree $\chi_0(1) = 2p - 1$, such that for this pair the assertion of the Theorem does not hold.

■

3 Preliminary lemmas

In the proof of Theorem 2.1 we will need the following lemma, which is Lemma 2 in [4].

Lemma 31. *Let G be a group with $H \in \mathrm{Hall}_\pi(G)$ and with the property that every π-subgroup Y of G can be conjugated into H. Let $\mathcal{K}$ be a class of elements of H, which is closed under conjugation inside H with elements of G such that if two elements of $\mathcal{K}$ are conjugate in G then they are already conjugate in H. Then if $G_1 \triangleleft G$ and $|G : G_1| = q$, where $q \in \pi$, then for $H_1 = H \cap G_1$ it holds that each pair of elements of $H_1 \cap \mathcal{K}$ that are conjugate in G_1 are already conjugate in H_1.* ■

For the proof of Theorem 2.1 we will also need the following lemma, which generalizes both Lemma 5 in [4] and Lemma 3.2 in [3].

Lemma 32. *Let $q \in \pi(G)\backslash\{p\}$ be a fixed prime, $P \in \mathrm{Syl}_p(G)$, $U < P$ abelian and strongly closed in P. Then if $(p, q) \not\leq \mathrm{N}_G(U)$ then $(p, q) \not\leq G$, as well.*

Proof. Let G be a counterexample of minimal order.

First we prove that we may assume that $\mathrm{O}_p(G) = 1$.

Let $\mathrm{O}_p(G) > 1$ and let $B \triangleleft G$ be a p-subgroup. Let $\overline{G} = G/B$, and the images of U and P in this factor group let $\overline{U}$ and $\overline{P}$, respectively. Then $\overline{P} \in Syl_p(\overline{G})$ and the triple $(\overline{G}, \overline{P}, \overline{U})$ satisfies the conditions set for (G, P, U). To see this we have to show only that $(p, q) \not\leq \mathrm{N}_G(U)$ implies $(p, q) \not\leq \mathrm{N}_{\overline{G}}(\overline{U})$. Let M be the inverse image of $\mathrm{N}_{\overline{G}}(\overline{U})$ in G. Here $M < G$, since if $\overline{U} \triangleleft \overline{G}$ then $U^G \leq P$, and as $U \leq P$ is strongly closed, $U^G = U$ would follow. This would imply $(p, q) \not\leq \mathrm{N}_G(U) = G$, which is a contradiction. So $M < G$. But $P \leq M$ and $\mathrm{N}_G(U) = \mathrm{N}_M(U)$. The triple (M, P, U) satisfies the conditions of the Lemma. By induction $(p, q) \not\leq M$. By [2], $(p, q) \not\leq \overline{M} = \mathrm{N}_{\overline{G}}(\overline{U})$, as well. Hence the conditions of the Lemma are satisfied by the triple $(\overline{G}, \overline{P}, \overline{U})$ and by induction $(p, q) \not\leq \overline{G}$.

Let V be a (p, q)-group in G. Then $V' = V_p \in \mathrm{Syl}_p(V)$ and by the above result, its image $\overline{V}$ in $\overline{G}$ is nilpotent. Hence $V_p \leq B$. There are two cases:

(i) $U \cap \mathrm{O}_p(G) = 1$,

(ii) $U \cap \mathrm{O}_p(G) \neq 1$.

Ad (i): We know that $U \triangleleft P$ and we may assume that $V_p \leq P$, by replacing V with a suitable conjugate of it. Hence $[U, V_p] \leq U$. On the other hand as $V_p \leq \mathrm{O}_p(G)$, $[U, V_p] \leq \mathrm{O}_p(G) \cap U = 1$.

Ad (ii): If $U \cap \mathrm{O}_p(G) \neq 1$, then we may choose B equal to it, because $U \cap \mathrm{O}_p(G)$ is normal in G as U is strongly closed. Hence $V_p \leq B \leq U$ by the above results. But U is abelian, so $[V_p, U] = 1$ in this case, too.

Hence in both cases (i) and (ii) $[V_p, U] = 1$. Let $N = \mathrm{N}_G(V_p)$. We claim that $N = G$. If $N < G$ then if we choose $S \in \mathrm{Syl}_p(N)$ with the property $U \leq S$, then the triple (N, S, U) satisfies the conditions of the Lemma. So by induction $(p, q) \not\leq N$. This contradicts the fact that $V \leq N$. Thus V_p is normal in G. Let $C = \mathrm{C}_G(V_p)$. Then $C \triangleleft G$. Let us choose $Q \in \mathrm{Syl}_q(G)$ so that it should contain a Sylow q-subgroup of V. Set $L = CQ$. Then $P \cap L = P \cap C \in \mathrm{Syl}_p(L)$ and $P \cap C \triangleleft P$. Then the triple $(L, P \cap L, U)$ satisfies the conditions of the Lemma. Hence if $L < G$ then by induction $(p, q) \not\leq L$, which is impossible as $V \leq L$. Thus $G = L = CQ$. Since $P = P \cap L = P \cap C$, $P \leq C$ and as $C \triangleleft G$, so by the Frattini argument we have that $G = C\mathrm{N}_G(P)$. As U is a strongly closed subgroup of P, $\mathrm{N}_G(P) \leq \mathrm{N}_G(U)$ and thus $(p, q) \not\leq \mathrm{N}_G(P)$. Since $V_p \leq \mathrm{C}_G(P) \leq \mathrm{N}_G(P)$ and $V_p \triangleleft G$, hence $V_p \triangleleft \mathrm{N}_G(P)$. Thus $|\mathrm{N}_G(P) : C \cap \mathrm{N}_G(P)| \not\equiv 0 \ (q)$. But then, since $|\mathrm{N}_G(P) : C \cap \mathrm{N}_G(P)| = |C\mathrm{N}_G(P) : C| = |G : C| = |CQ : C| = |Q : C \cap Q|$, $Q \leq C = G$ follows. This is a contradiction, since $V \leq G = \mathrm{C}_G(V_p)$.

End of the proof: Let $\mathcal{A} = \{A|\ \mathrm{N}_G(A) < G, A\ p\text{-}group,\ (p, q) \leq \mathrm{N}_G(A)\}$. Let K be a maximal element of $\mathcal{A}$ for the Thompson-ordering. Then $|\mathrm{N}_G(K)|_p$ is maximal and among those with this property K is also of maximal order. As $\mathcal{A} \neq \emptyset$, so such K exists. Then $K \leq P_1$ for a suitable Sylow p-subgroup P_1 of G. Let $x \in G$ such that $P_1^x = P$. Then $K^x \leq P$. Let $\mathrm{Z}(P_1) \leq R \in \mathrm{Syl}_p(\mathrm{N}_G(K))$. Then $\mathrm{Z}(P) \leq R^x \in \mathrm{Syl}_p(\mathrm{N}_G(K^x))$. Let $R^x \leq P_2 \in \mathrm{Syl}_p(G)$. Choose $t \in G$ so that $P_2^t = P$. Thus $R^{xt} \leq P$. Since $U \triangleleft P$, $\mathrm{Z}(P) \cap U \neq 1$, thus $R^x \cap U \neq 1$, as well. As U is stongly closed in P, $(R^x \cap U)^t \leq R^{xt} \cap U$ and $R^{xt} \cap U$ is strongly closed in R^{xt}. It is enough to prove that the triple $(\mathrm{N}_G(K^{xt}), R^{xt}, R^{xt} \cap U)$ satisfies the conditions of the Lemma. When we prove this, then $(p, q) \not\leq \mathrm{N}_G(K^{xt})$ follows, contradicting our assumption. It is enough to prove that $(p, q) \not\leq \mathrm{N}_G(R^{xt} \cap U)$. If $|R| = |P|$ then we have that the triple $(\mathrm{N}_G(K^{xt}), P, U)$ satisfies the conditions of the Lemma, and since $\mathrm{N}_G(K^{xt}) < G$, by induction we get that $(p, q) \not\leq \mathrm{N}_G(K^{xt})$, contradicting the choice of K. Thus $|R| < |P|$. Then $\mathrm{N}_P(R^{xt}) > R^{xt}$, and since $R^{xt} \cap U$ is strongly closed in R^{xt}, $\mathrm{N}_G(R^{xt}) \leq \mathrm{N}_G(R^{xt} \cap U)$.

So $|\mathrm{N}_G(R^{xt} \cap U)|_p > |\mathrm{N}_G(K^{xt})|_p = |R|$. As $\mathrm{N}_G(R^{xt} \cap U) \neq G$, thus $(p,q) \not\lesssim \mathrm{N}_G(R^{xt} \cap U)$, by the maximality of K in the Thompson ordering. The proof is complete. ■

For the proof of Theorem 2.4 we will need the description of irreducible characters of minimal non-nilpotent groups. As we did not find any reference to it in the literature, for the sake of selfcontainedness we include it here.

Lemma 33. *Let G be a (p,q)-group, $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$, $|Q| = q^n$. Then G has exactly q^n linear characters.*

(i) *If P is abelian, then all other characters in $\mathrm{Irr}(G)$ are of degree q. They are induced from nontrivial characters of the unique index q subgroup of G. There are $(|P|-1)q^{n-2}$ such characters.*

(ii) *If P is extraspecial, then $|P| = p^{2m+1}$, where $2m \equiv o(p)(\mathrm{mod}(q))$. $P/\mathrm{Z}(P)Q$ is a (p,q)-group of type (i). So it has $(p^{2m}-1)q^{n-2}$ irreducible characters of degree q. The $p-1$ irreducible characters of degree p^m of P can be extended to G giving $(p-1)q^n$ irreducible characters of degree p^m.*

(iii) *If P is special and nonabelian, then if $|\mathrm{Z}(P)| = p^k$ then $\mathrm{Z}(P)$ has $\frac{p^k-1}{p-1}$ maximal subgroups. By factoring with one of them we get a (p,q)-group of type (ii). The union of inverse images of these characters give $\mathrm{Irr}(G)$.*

Proof. As $G' = P$, $|G : G'| = q^n$, so G has exactly q^n linear characters. Let $H = P\langle x^q \rangle$. Then $|G : H| = q$ and H is normal in G.

Ad (i): If P is abelian, then so is H, so if $\chi \in \mathrm{Irr}(G)$ nonlinear, then $\chi_H = \sigma_1 + ... + \sigma_q$ and $\chi = \sigma_i^G$ for $i = 1, ..., q$. So $\chi(1) = q$ and χ is induced from exactly q linear characters of H. As $|G| = |P|q^n = q^n + q^2(|P|-1)q^{n-2}$, we get that each nontrivial character of H that does not contain P in its kernel is induced to $\mathrm{Irr}(G)$.

Ad (ii): If P is extraspecial, then $|P| = p^{2m+1}$. As Q acts irreducibly on $P/\mathrm{Z}(P)$, by Lemma 3.10 in Chapter II. of [6] we get that $2m = \mathrm{o}(p)(\mathrm{mod}(q))$. The $p-1$ faithful irreducible characters of P are of degree p^m, they are 0 outside $\mathrm{Z}(P)$, so they are G-invariant, and as $(|P|, |G : P|) = 1$, they can be extended to G. By Gallagher's theorem, se e.g. [7], they can be extended in q^n ways. This way we get $(p-1)q^n$ irreducible characters of G. By taking into consideration those of degree 1 and q the sum of squares of the degrees gives:

$$q^n + q^2(p^{2m} - 1)q^{n-2} + p^{2m}(p-1)q^n = q^n p^{2m+1} = |G|,$$

so we determined all $\mathrm{Irr}(G)$.

Ad (iii): We calculate the sum of squares of the irreducible characters we produced so far: $q^n+q^2(p^{2m}-1)q^{n-2}+p^{2m}\frac{p^k-1}{p-1}(p-1)q^n = q^n p^{2m+k} = |G|$, so we produced the whole $\mathrm{Irr}(G)$. ■

4 Proofs of the main results

Now we prove Theorem 2.1:

***Proof of Thorem* 2.1.** (i) → (ii) is trivial.

(ii) → (iii): We use induction on $|G|$. The following argument is similar to one in the last part of the proof of Theorem 1 in [4]. For the sake of selfcontainedness, we repeat it here.

Let $A \triangleleft P$ be an abelian normal subgroup in P such that $\exp(A) \leq p$ if $p > 2$, and $\exp(A) \leq 4$ if $p = 2$, and A is maximal with these properties.

(a) If $A \leq \mathrm{Z}(P)$:

then according to Alperin's theorem [1], $\Omega_j(P) \leq \mathrm{Z}(P)$, where $j = 1$ if $p > 2$ and $j = 2$ if $p = 2$. Then A is strongly closed in P, as if we take two elements a and a^x of order p or of order 4 in A and P, then they are conjugate in P, and as A is normal in P, we get that $a^x \in A$, too. Let $N = \mathrm{N}_G(A)$. If $N < G$, then as $P \in \mathrm{Syl}_p(N)$, by induction we get that $(p, q) \not\lesssim N$, and by Lemma 3.2 , $(p, q) \not\lesssim G$. So we may assume that $\mathrm{N}_G(A) = G$. Then $P \leq \mathrm{C}_G(A) \triangleleft G$. As for each $a \in A^{\#}$ for the conjugacy class $\mathrm{K}_G(a)$ of a in G and for the conjugacy class $\mathrm{K}_P(a)$ in P it holds that $\mathrm{K}_G(a) = \mathrm{K}_G(a) \cap P = \mathrm{K}_P(a) = a$, as $A \triangleleft G$ and $A \leq \mathrm{Z}(P)$, so $|G : \mathrm{C}_G(a)| = |P : \mathrm{C}_P(a)| = 1$, and we get that $A \leq \mathrm{Z}(G)$. Thus, if $p > 2$, then $\Omega_1(P) \leq A \leq \mathrm{Z}(G)$; if $p = 2$, then $\Omega_2(P) \leq \mathrm{Z}(G)$; and this means that $G \not\gtrsim (p, q)$ in cases $p > 2$ and $p = 2$, either, for every prime divisor $q \neq p$ of G.

(b) If $A \not\leq \mathrm{Z}(P)$:

then $A \cap \mathrm{Z}(P) < A$. Thus $A/A \cap \mathrm{Z}(P)$ contains a central subgroup of $P/A \cap \mathrm{Z}(P)$ of order p. Let A_1 be its inverse image in A. According to our assumption, $A_1 = \langle A \cap \mathrm{Z}(P), x\rangle$, where $o(x) = p$ or $o(x) = 4$ and $x \notin \mathrm{Z}(P)$. A_1 is strongly closed in P as if $a_1 \in A_1$ and $a_1^u \in P$, then by assumption a_1^u is conjugate to a_1 in P. But as $A_1 \triangleleft P$, $a_1^u \in A_1$. If

$N_1 = \mathrm{N}_G(A_1) < G$, then as $P \leq N_1$, by induction we have that $(p,q) \not\leq N_1$. Then, by Lemma 3.2, $(p,q) \not\leq G$. So, we may assume that $N_1 = G$. Then $\mathrm{C}_G(A_1) \lhd G$ and $A \cap \mathrm{Z}(P) \leq \mathrm{Z}(G)$, as if $a \in A \cap \mathrm{Z}(P)$ and $g \in G$ then $a^g \in A_1 \leq P$, so by assumption there is a $u \in P$ such that $a^g = a^u$ and as $a \in A \cap \mathrm{Z}(P)$ $a^u = a$. So, $|G : \mathrm{C}_G(a)| = 1$ and thus $A \cap \mathrm{Z}(P) \leq \mathrm{Z}(G)$ and $\mathrm{C}_G(A_1) = \mathrm{C}_G(x)$. Let $g \in G$, then $x^g \in A_1 \leq P$, so there exists an $u \in P$ such that $x^g = x^u$, therefore the conjugacy classes $\mathrm{K}_G(x)$ and $\mathrm{K}_P(x)$ coincide and thus $|G : \mathrm{C}_G(A_1)| = |G : \mathrm{C}_G(x)| = |P : \mathrm{C}_P(x)| > 1$. So, $\mathrm{C}_G(A_1)$ is a proper normal subgroup of p-power index in G and it is contained in some normal subgroup G_1 of index p. By Lemma 3.1 , applied for $H = P \in \mathrm{Syl}_p(G)$ if we take $\mathcal{K}$ to be the set of elements of P of order p or 4, then induction gives that $(p,q) \not\leq G_1$ for every prime $q \neq p$. As a (p,q)-group is generated by its Sylow q-subgroups, $(p,q) \not\leq G$, either.

(iii) $\rightarrow$ (iv): Let T be a p-subgroup of G. Let $N = \mathrm{N}_G(T)$. Let $q \in \pi(N)\backslash\{p\}$, $Q \in \mathrm{Syl}_q(N)$. If $[Q,T] \neq 1$, then $(p,q) \leq QT$, which cannot happen by assumption. Hence (iv) follows.

(iv) $\rightarrow$ (i): The proof is similar to the second part of Lemma 5 in [4]. For the sake of selfcontainedness we repeat it here.

Let $a, b \in P$ such that $a = b^x$ for some $x \in G$. By the thereom of Alperin, see e.g. Chapter 7, Theorem 2.6 in [5], there exist Sylow p-subgroups $Q_1, ..., Q_n$ of G, elements $x_1, ..., x_n$ with $x_j \in \mathrm{N}_G(P \cap Q_j)$, and $y \in \mathrm{N}_G(P)$ such that $b \in P \cap Q_1$, $b^{x_1 \dots x_{j-1}} \in P \cap Q_j$, $x = x_1 ... x_n y$ and $\mathrm{N}_P(P \cap Q_j) \in \mathrm{Syl}_p(N_G(P \cap Q_j))$ for $j = 1, ..., n$. Let $N_j = \mathrm{N}_G(P \cap Q_j)$, $C_j = \mathrm{C}_G(P \cap Q_j)$, $P_j = \mathrm{N}_P(P \cap Q_j)$, $j = 1, ..., n$.

So $N_j = C_j(P \cap N_j)$ and hence $x_j = y_j z_j$, where $y_j \in C_j$ and $z_j \in P \cap N_j$. It is easy to see that that $a = b^x = b^{z_1 \dots z_n y}$. $\mathrm{N}_G(P) = \mathrm{C}_G(P)P$, so $y = c z_{n+1}$, where $c \in \mathrm{C}_G(P)$, $z_{n+1} \in P$. Thus $a = b^x = b^{z_1 \dots z_{n+1}}$, which means that a and b are conjugate in P. The proof is complete. ∎

Now we prove Theorem 2.2:

***Proof of Theorem* 2.2.** Let $H = \mathrm{O}_{p',p}(G)$, $R = H \cap P$. As G is p-constrained, $\mathrm{C}_G(R) \leq H$. By the Frattini argument, $G = \mathrm{O}_{p'}(G)\mathrm{N}_G(R)$. Let $q \neq p$ prime, $Q \in \mathrm{Syl}_q(\mathrm{N}_G(R))$. Then $QR = Q \times R$, as $(p,q) \not\leq QR$. Hence $Q \leq \mathrm{C}_G(R) \leq H$, and so $Q \leq \mathrm{O}_{p'}(G)$. Thus $G = \mathrm{O}_{p'}(G)P$. ∎

Now we prove Theorem 2.3:

***Proof of Theorem* 2.3.** (i) $\rightarrow$(ii): Repeating the argument of the previous proof one gets that $\mathrm{O}^{q'}(G) \leq \mathrm{O}_{p'}(G)$. As $\mathrm{O}^{q'}(G) \lhd G$ and it is a p'-group, so $\mathrm{O}^{q'}(G)P$ is a subgroup of G having normal p-complement.

(ii)→(i): If $O^{q'}(G)P$ has a normal p-complement, then $O^{q'}(G)$ is a p'-subgroup. If U is a (p,q)-group in G, then $U \leq O^{q'}(U) \leq O^{q'}(G)$. As $O^{q'}(G)$ is a p'-subgroup, this is a contradiction. So, the proof is complete. ∎

Remark 41. In Theorem 2.3 the condition that G is p-constrained cannot be omitted. Take $G = A_5$, $P \in \mathrm{Syl}_5(G)$. Then $(5,3) \not\leq G$, since 25 does not divide $|G|$. As G is simple, $G = O^{3'}(G) = O^{3'}(G)P$.

Now we prove Theorem 2.4:

***Proof of Theorem* 2.4.** We use induction on $|G|$. We assume that G is a finite p-constrained group of minimal order that has character $\chi \in \mathrm{Char}(G)$ satisfying the conditions in our Theorem such that the assertion is not yet known. We may assume that (ii) is false for G, and we have to prove that for G (i) holds. As (ii) does not hold for G, so it cannot hold for any proper subgroup H of G. So by induction (i) holds for all such H. Using α) and β) we can deduce that every constituent of χ_P is linear, and thus $P' \leq \mathrm{Ker}(\chi)$. By γ), P is abelian. As G is p-constrained, so $P' = 1$ implies that G is p-solvable. By the choice of G we get immidiately that $\pi(G) = \{p,q\}$ for a suitable prime $q \neq p$. This gives that G is solvable, hence G is also q-constrained.

We have to prove that $P \triangleleft G$. As $\pi(G) = \{p,q\}$, this means that G is q-nilpotent. As G is q-constrained, our Theorem 2.1 (iii) and Theorem 2.2 implies, (even without the use of the transfer), that either $P \triangleleft G$ or G is a (q,p)-group. To finish the proof, it is enough to show that the second possibility cannot occur. Assume that G is a (q,p)-group. Let m be $o(q) \pmod{(p)}$. Using β) an appeal to Lemma 33. shows that χ can have a nonlinear constituent only in the case when m is even, say $m = 2a$. The degree of a nonlinear irreducible constituent of χ is then q^a. Since $q^a + 1 \equiv 0 \pmod{(p)}$, $q^a = pl - 1$, for a suitable natural number l. From this one deduces that either $q^a \geq 2p - 1$ or $q^a = p - 1$ and $q = 2$. If $q^a = p - 1$, $p = 2$ cannot occur. As (ii) is not true in G, case $q^a = p-1$ cannot hold, either. So χ has only linear constituents. But then $G' \leq \mathrm{Ker}(\chi)$, so by γ) G has to be abelian in this case, contradicting the assumption that G is a (q,p)-group. This completes the proof. ∎

Now we give an example showing that in α) $2p - 2$ cannot be replaced by $2p - 1$.

Let p and $q = 2p - 1$ be primes, where $p \geq 7$. E.g. $p = 7$ and $q = 13$. Let G_0 be a (q,p)-group of order q^3p with extraspecial Sylow q-subgroup. In this case $o(q)(\mathrm{mod} p) = 2$, so such a group exists. Then $Q_0 = G_0' \in \mathrm{Syl}_q(G_0)$, and G_0 has a character χ_0 of degree q which is irreducible and faithful. So α) is not satisfied for χ_0, as $\chi_0(1) = 2p-1$. Since χ_0 is faithful γ) holds. As all proper subgroups of G_0, except for Q_0, are abelian, and χ_{Q_0} is irreducible, for χ_0 β) also holds. (i) is not true for G_0. But p is not a Fermat prime either, as then $p=2^{2^k}+1$ would hold and $q=2p-1=2^{2^k+1}+1\equiv 0(\mathrm{mod}(3))$ so (ii) cannot hold, either.

Remark 42. This theorem extends a well-known result of N. Itô ([8], see also [7]).

It can be deduced from our statement if we replace $2p-2$ by $p-1$ in α) and we assume also γ). Then β) is automatically satisfied. If P is not normal in G, then $\chi(1) = p-1$ and $\chi \in \mathrm{Irr}(G)$ also holds.

On the other hand the assumption β) is vital for our proof as if case (i) holds, then, by a theorem of N. Itô (see e.g. [7]), $(\chi(1), p) = 1$ holds for every irreducible character $\chi \in \mathrm{Irr}(G)$.

Remark 43. The conditions of our Theorem 2.4 however do not guarantee that if P is not normal in G, then χ should be irreducible. Let us take $p = 2^{2^k}+1$ to be a Fermat-prime. Let G be a $(2,p)$-group of order $2^{2^{k+1}+1}p$ with extraspecial Sylow 2-subgroup and Sylow p-subgroup P of order p. Then G has a faithful irreducible character χ of degree $p-1$. Let us choose a character $\sigma \in \mathrm{Char}(G)$ with $p-1 < \sigma(1) \leq 2p-2$, and $(\sigma,\chi) = 1$ and all other constituents of σ are choosen to be linear. Then σ satisfies $\alpha), \beta)$ and γ), but σ is not irreducible.

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References

[1] J.L. Alperin, *Centralizers of abelian normal subgroups of p-groups*, J. Algebra **1** (1964), 110–113.

[2] K. Corrádi, *On certain properties of centralizers hereditary to the factor group*, Publ. Math. (Debrecen) **37** (1990), 203–206.

[3] K. Corrádi and E. Horváth, *Steps towards an elementary proof of Frobenius' theorem*, Comm. Algebra **24** (1996), 2285–2292.

[4] K. Corrádi and E. Horváth, *Normal π-complement theorems*, Arch. Math. (Basel) **71** (1998), 262–269.

[5] D. Gorenstein, "*Finite groups*", Chelsea Publ. Comp., New York 1980.

[6] B. Huppert, "*Endliche Gruppen*", Springer-Verlag, Berlin 1967.

[7] I.M. Isaacs, "*Character theory of finite groups*", Dover Publ., Inc., New York 1994.

[8] N. Itô, *On a theorem of H.F. Blichfeldt*, Nagoya Math. J. **5** (1953), 75–77.

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