

RING-LIKE OPERATIONS IN PSEUDOCOMPLEMENTED SEMILATTICES

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Abstract

Ring-like operations are introduced in pseudocomplemented semilattices in such a way that in the case of Boolean pseudocomplemented semilattices one obtains the corresponding Boolean ring operations. Properties of these ring-like operations are derived and a characterization of Boolean pseudocomplemented semilattices in terms of these operations is given. Finally, ideals in the ring-like structures are defined and characterized.

Keywords: pseudocomplemented semilattice, Boolean algebra, Boolean ring, distributivity, linear equation, ideal, congruence kernel.

1991 Mathematics Subject Classification: 06A12, 08C10, 06E20, 16Y99.

¹This paper is a result of the collaboration of the authors within the framework of the “Aktion Österreich-Tschechische Republik” (grant No. 22p2 “Ordered algebraic structures and applications”).

1. INTRODUCTION

First we will briefly report on some results (obtained in [2] and [5]) concerning ring-like operations in orthomodular lattices since ring-like operations will be introduced in pseudocomplemented semilattices in a similar way and we will obtain also similar results.

It is well-known that there is a natural bijection between Boolean algebras and Boolean rings. This correspondence between certain lattice structures and certain term-equivalent ring structures is very useful. For instance, congruence permutability of Boolean algebras follows immediately from that of rings (resp. groups). So it is natural to ask how a ring-like structure can be introduced in generalizations of Boolean algebras. In [1] and [3] a ring-like structure was introduced in orthomodular lattices and also in more general structures. Since pseudocomplemented semilattices can also be viewed as generalizations of Boolean algebras one may try to introduce a ring-like structure in pseudocomplemented semilattices. This is the aim of the present paper.

2. RING-LIKE OPERATIONS IN ORTHOMODULAR LATTICES

An *orthomodular lattice* is an algebra $(L, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and additionally we have:

- (i) $(x')' = x$,
- (ii) $(x \vee y)' = x' \wedge y'$,
- (iii) $x \vee x' = 1$,
- (iv) $x \leq y \Rightarrow y = x \vee (y \wedge x')$,

for all $x, y \in L$.

In the following let L be an arbitrary, fixed orthomodular lattice. On every Boolean subalgebra of L one can define ring operations $+$ and \cdot by

$$\begin{aligned} x + y &:= (x \wedge y') \vee (x' \wedge y) = (x \vee y) \wedge (x' \vee y'), \\ xy &:= x \wedge y = (x \vee y) \wedge (x \vee y') \wedge (x' \vee y). \end{aligned}$$

We now extend these operations from the Boolean subalgebras of L to L by defining

$$\begin{aligned} x +_1 y &:= (x \wedge y') \vee (x' \wedge y), \\ x +_2 y &:= (x \vee y) \wedge (x' \vee y'), \\ x \cdot_1 y &:= x \wedge y, \\ x \cdot_2 y &:= (x \vee y) \wedge (x \vee y') \wedge (x' \vee y), \end{aligned}$$

for all $x, y \in L$. Then the following theorem holds:

Theorem 1. *For arbitrary, fixed $i, j \in \{1, 2\}$ the following properties are equivalent:*

- (i) L is a Boolean algebra;
- (ii) $+_1 = +_2$;
- (iii) $\cdot_1 = \cdot_2$;
- (iv) $+_i$ is associative;
- (v) \cdot_2 is associative;
- (vi) \cdot_j is distributive with respect to $+_i$;
- (vii) For every $(a, b) \in L^2$ the equation $a +_i x = b$ has at most one solution;
- (viii) For every $(a, b) \in L^2$ the equation $a +_i x = b$ has exactly one solution;
- (ix) For every $(a, b) \in L^2$ the equation $a +_i x = b$ has at least one solution.

Proof. See [2] and [5]. ■

3. RING-LIKE OPERATIONS IN PSEUDOCOMPLEMENTED SEMILATTICES

A *pseudocomplemented meet-semilattice* (with zero) is an algebra $S = (S, \wedge, *, 0)$ of type $(2, 1, 0)$, where $(S, \wedge, 0)$ is a meet-semilattice with smallest element 0 and where each $x \in S$ has a so-called pseudocomplement x^* , that is a greatest element $y \in S$ with the property $x \wedge y = 0$, i. e. for $x, y \in S$ it holds $x \wedge y = 0$ iff $y \leq x^*$. (The concept of a *pseudocomplemented join-semilattice* (with one) can be defined dually.)

In this section let S denote an arbitrary, fixed pseudocomplemented meet-semilattice and let a, b, c be arbitrary, fixed elements of S .

Put $a \sqcup b := (a^* \wedge b^*)^*$ and $1 := 0^*$.

The following facts are well-known (cf. [4]):

- (i) $(*, *)$ is a Galois correspondence between (S, \leq) and (S, \leq) ,
- (ii) $a \leq b \Rightarrow a^* \geq b^*$,
- (iii) $a \leq a^{**}$,

- (iv) $a^{***} = a^*$,
- (v) $0^{**} = 0$,
- (vi) $** \in \text{End } S$;
- (vii) $\theta_0 := \ker^{**} \in \text{Con}S$,
- (viii) $BA(S) := (S^*, \sqcup, \wedge, *, 0, 1)$ is the greatest Boolean subalgebra of $(S, \sqcup, \wedge, *, 0, 1)$.

We call $BA(S)$ the *Boolean algebra induced by S* .

Remark. (i) $\theta_0 \in \text{Con}(S, \sqcup, \wedge, *, 0, 1)$ and from the homomorphism theorem it follows that $(S, \sqcup, \wedge, *, 0, 1)/\theta_0 \cong BA(S)$.

(ii) One can show that the class of all pseudocomplemented meet-semilattices forms a variety which can be defined by the following laws:

- (1) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$,
- (2) $x \wedge y = y \wedge x$,
- (3) $x \wedge x = x$,
- (4) $x \wedge 0 = 0$,
- (5) $(x \wedge y)^* \wedge (x \wedge y^*)^* = x^*$,
- (6) $0^{**} = 0$,
- (7) $x \wedge x^{**} = x$.

Let $BR(S) := (S^*, +, 0, \cdot, 1)$ denote the Boolean ring corresponding to the Boolean algebra $BA(S)$. We call $BR(S)$ the *Boolean ring induced by S* . Then

$$\begin{aligned} a + b &= (a \wedge b^*) \sqcup (a^* \wedge b) = (a \sqcup b) \wedge (a^* \sqcup b^*), \\ ab &= a \wedge b = (a \sqcup b) \wedge (a \sqcup b^*) \wedge (a^* \sqcup b), \end{aligned}$$

if $a, b \in S^*$. We now extend these operations from S^* to S in several "natural" ways by defining

$$\begin{aligned} a + b &:= a^{**} + b^{**}, \\ a +_1 b &:= (a \wedge b^*) \sqcup (a^* \wedge b), \\ a +_2 b &:= (a \sqcup b) \wedge (a^* \sqcup b^*), \\ ab &:= a^{**}b^{**}, \\ a \cdot_1 b &:= a \wedge b, \\ a \cdot_2 b &:= (a \sqcup b) \wedge (a \sqcup b^*) \wedge (a^* \sqcup b). \end{aligned}$$

Since S^* is a subalgebra of S and $S \sqcup S \subseteq S^*$, we have $S + S$, $S +_1 S$, $S +_2 S$, SS , $S \cdot_2 S \subseteq S^*$.

Lemma 2. $+ = +_1 = +_2$ and $\cdot = \cdot_2$

Proof. For $i \in \{1, 2\}$ it holds

$$\begin{aligned} a +_i b &= (a +_i b)^{**} = a^{**} +_i b^{**} = a^{**} + b^{**} = a + b, \\ a \cdot_2 b &= (a \cdot_2 b)^{**} = a^{**} \cdot_2 b^{**} = a^{**} b^{**} = ab. \end{aligned} \quad \blacksquare$$

Obviously, $+$, \cdot and \cdot_1 are commutative and associative and \cdot is distributive with respect to $+$.

To S , we assign the following ring-like structures: $R(S) := (S, +, 0, \cdot, 1)$ and $P(S) := (S, +, 0, \cdot_1, 1)$.

From the homomorphism theorem, it follows that

$$\theta_0 \in \text{Con}R(S) \cap \text{Con}P(S) \text{ and } R(S)/\theta_0 \cong P(S)/\theta_0 \cong BR(S).$$

Lemma 3. *The following identities hold:*

$$\begin{array}{lll} a + 0 = a^{**}, & a0 = 0, & a \cdot_1 0 = 0, \\ a + a = 0, & aa = a^{**}, & a \cdot_1 a = a, \\ a + a^* = 1, & aa^* = 0, & a \cdot_1 a^* = 0, \\ a + a^{**} = 0, & aa^{**} = a^{**}, & a \cdot_1 a^{**} = a, \\ a + 1 = a^*, & a1 = a^{**}, & a \cdot_1 1 = a. \end{array}$$

Proof. Straightforward. ■

Lemma 4. $(a + b)^* = a^* + b = a + b^*$.

Proof. Indeed, we have

$$\begin{aligned} (a + b)^* &= (a + b) + 1 = (a + 1) + b = a^* + b, \\ (a + b)^* &= (a + b) + 1 = a + (b + 1) = a + b^*. \end{aligned} \quad \blacksquare$$

Corollary. $a + b = (a + b)^{**} = (a^* + b)^* = a^* + b^*$. ■

The following lemma characterizes vanishing of the symmetric difference $a + b$:

Lemma 5. *The following properties are equivalent:*

- (i) $a + b = 0$;
- (ii) $a\theta_0 b$;
- (iii) *There exists a $c \in S$ with $a + c = b + c$;*
- (iv) $a + x = b + x$ for all $x \in S$.

Proof. (i) \Rightarrow (ii): $a^{**} = a + 0 = a + (a + b) = (a + a) + b = 0 + b = b^{**}$.
(ii) \Rightarrow (iv): $a + x = a^{**} + x = b^{**} + x = b + x$ for all $x \in S$.
(iv) \Rightarrow (iii): Straightforward.
(iii) \Rightarrow (i): $a + b = (a + b) + 0 = (a + b) + (c + c) = (a + c) + (b + c) = (a + c) + (a + c) = 0$. ■

Lemma 6. *The following properties are equivalent:*

- (i) $a + b = 1$;
- (ii) $a\theta_0 b^*$;
- (iii) $a^*\theta_0 b$.

Proof. Indeed, we have

$$\begin{aligned} a + b = 1 &\Leftrightarrow (a + b)^* = 1^* \Leftrightarrow a + b^* = 0 \Leftrightarrow a\theta_0 b^*, \\ a + b = 1 &\Leftrightarrow (a + b)^* = 1^* \Leftrightarrow a^* + b = 0 \Leftrightarrow a^*\theta_0 b, \end{aligned}$$

according to Lemma 5. ■

Now we want to characterize Boolean pseudocomplemented semilattices. For this purpose we first need two lemmas. The first of these describes the set of all solutions of a linear equation:

Lemma 7. *We have:*

$$\{x \in S \mid a + x = b\} = \begin{cases} \emptyset & \text{if } b \notin S^*, \\ [a + b]\theta_0 & \text{if } b \in S^*. \end{cases}$$

Proof. The first part follows from $S + S \subseteq S^*$. In order to prove the second part, assume $b \in S^*$ and let $x \in S$. Then

$$\begin{aligned} a + x = b &\Leftrightarrow a^{**} + x^{**} = b \Leftrightarrow x^{**} = a^{**} + b \Leftrightarrow x^{**} = \\ &= (a + b)^{**} \Leftrightarrow x\theta_0 a + b \Leftrightarrow x \in [a + b]\theta_0. \end{aligned} \quad \blacksquare$$

Proposition 8. *The following properties hold:*

- (i) $ab = a \cdot_1 b \Leftrightarrow a^{**} \wedge b^{**} \leq a \wedge b$;
- (ii) $(a + b) \cdot_1 c = (a \cdot_1 c) + (b \cdot_1 c) \Leftrightarrow (a + b) \wedge c^{**} \leq c$;
- (iii) $|\{x \in S \mid a + x = b\}| \leq 1 \Leftrightarrow (b \notin S^* \text{ or } |[a + b]\theta_0| = 1)$;
- (iv) $|\{x \in S \mid a + x = b\}| = 1 \Leftrightarrow (b \in S^* \text{ and } |[a + b]\theta_0| = 1)$;
- (v) $|\{x \in S \mid a + x = b\}| \geq 1 \Leftrightarrow b \in S^*$.

Proof. (i) is evident.

Ad (ii): $(a + b) \cdot_1 c = (a + b) \wedge c$ and

$$\begin{aligned} (a \cdot_1 c) + (b \cdot_1 c) &= (a \wedge c) + (b \wedge c) = ((a \wedge c) + (b \wedge c))^{**} = \\ &= (a^{**} \wedge c^{**}) + (b^{**} \wedge c^{**}) = \\ &= (a^{**} + b^{**}) \wedge c^{**} = (a + b) \wedge c^{**}. \end{aligned}$$

Hence,

$$\begin{aligned} (a + b) \cdot_1 c = (a \cdot_1 c) + (b \cdot_1 c) &\Leftrightarrow (a + b) \wedge c = (a + b) \wedge c^{**} \Leftrightarrow \\ &\Leftrightarrow (a + b) \wedge c^{**} \leq (a + b) \wedge c \Leftrightarrow \\ &\Leftrightarrow ((a + b) \wedge c^{**} \leq a + b \\ \text{and } (a + b) \wedge c^{**} \leq c &\Leftrightarrow (a + b) \wedge c^{**} \leq c. \end{aligned}$$

The properties (iii) – (v) follow from Lemma 7. ■

Now we are ready to prove the result concerning the characterization of Boolean pseudocomplemented semilattices in terms of ring-like operations.

Theorem 9. *The following are equivalent:*

- (i) S is a Boolean algebra;
- (ii) $\cdot = \cdot_1$;
- (iii) \cdot_1 is distributive with respect to $+$;
- (iv) The equation $a + x = b$ has at most one solution;
- (v) The equation $a + x = b$ has exactly one solution;
- (vi) The equation $a + x = b$ has at least one solution.

Proof. Obviously, (i) \Rightarrow (ii) – (vi) and (v) \Rightarrow (iv), (vi).

(ii) \Rightarrow (i): According to (i) of Proposition 8, $a^{**} \wedge b^{**} \leq a \wedge b$. Hence $a \leq a^{**} = a^{**} \wedge 1^{**} \leq a \wedge 1 = a$ which implies $a = a^{**}$.

(iii) \Rightarrow (i): According to (ii) of Proposition 8, $(a + b) \wedge c^{**} \leq c$. Hence, $c \leq c^{**} = (0 + 1) \wedge c^{**} \leq c$ and, therefore, $c = c^{**}$.

(iv) \Rightarrow (i): According to (iii) of Proposition 8, $|[a+b]\theta_0| = 1$ if $b \in S^*$. Since $c, c^{**} \in [0 + c^{**}]\theta_0$, we have $c = c^{**}$.
 (vi) \Rightarrow (i) follows from (v) of Proposition 8. ■

4. IDEALS IN GENERALIZATIONS OF PSEUDOCOMPLEMENTED SEMILATTICES

In the following let \mathcal{A} be an algebra such that there exists a binary operation \cdot on A , a unary operation $*$ on A as well as an element 0 of A such that $0x = xx^* = 0$ and $x0^* = x$ for all $x \in A$.

Every pseudocomplemented meet-semilattice $S = (S, \wedge, *, 0)$ can be considered as such an algebra. (Take $A := S$ and $\cdot := \wedge$.)

We now define the notions of an ideal of \mathcal{A} and of a congruence kernel of A , respectively. But first let us recall the notion of a unary polynomial function on A : A function p from A to A is called a *unary polynomial function* on A if there exists a positive integer n , an n -ary term function t on A and $a_2, \dots, a_n \in A$ with $p(x) = t(x, a_2, \dots, a_n)$ for all $x \in A$.

Definition 10. Let B be a non-empty subset of the algebra \mathcal{A} . B is called an *ideal of \mathcal{A}* – in signs $B \triangleleft A$ – if for all $a, b, c \in B$ and for every unary polynomial function p on A , $p(a)(p(b)c^*)^* \in B$. B is called a *congruence kernel of \mathcal{A}* if $B = [0]\theta$ for some $\theta \in \text{Con}A$.

The following theorem holds for arbitrary universal algebras:

Theorem 11. *A non-empty subset B of A is a class of some congruence on \mathcal{A} iff for every unary polynomial function p on A , $a, b, p(a) \in B$ implies $p(b) \in B$.*

Proof. See [6]. ■

Now, we are able to prove our final result concerning the fact that both notions defined in Definition 10 coincide:

Theorem 12. *The ideals of \mathcal{A} coincide with the congruence kernels of \mathcal{A} .*

Proof. Let $I \subseteq A$. First assume $I \triangleleft A$. Let $a, b \in I$, let p be a unary polynomial function on A and assume $p(a) \in I$. Then $p(b) = p(b)(p(a)(p(a))^*)^* \in I$. Hence, by Theorem 11, there exists some $c \in A$ and some $\theta \in \text{Con}(\mathcal{A})$ such that $I = [c]\theta$. Let q denote the unary zero polynomial function on A . Then $0 = q(c)(q(c)c^*)^* \in I$. Hence $I = [0]\theta$ which shows that I is a congruence kernel of \mathcal{A} .

Conversely, assume I to be a congruence kernel of \mathcal{A} . Then there exists some $\alpha \in \text{Con}(\mathcal{A})$ with $I = [0]\alpha$. Let $d, e, f \in I$ and let r be a unary polynomial function on A . Then $r(d)(r(e)f^*)^* \alpha r(0)(r(0)0^*)^* = 0$ which shows $r(d)(r(e)f^*)^* \in [0]\alpha = I$. Hence $I \triangleleft A$. ■

Remark. The ring-like structures $P(S)$ (where S is a pseudocomplemented meet-semilattice), Boolean quasirings (introduced in [3]) and orthopseudorings (introduced in [1]) are also special cases of the algebras considered in this section.

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Received 21 September 1998

Revised 7 June 1999