

## STRUCTURE THEOREMS FOR RIGHT *pp*-SEMI- GROUPS WITH LEFT CENTRAL IDEMPOTENTS\*

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### Abstract

The concept of strong spined product of semigroups is introduced. We first show that a semigroup  $S$  is a *rpp*-semigroup with left central idempotents if and only if  $S$  is a strong semilattice of left cancellative right stripes. Then, we show that such kind of semigroups can be described by the strong spined product of a *C-rpp*-semigroup and a right normal band. In particular, we show that a semigroup is a *rpp*-semigroup with left central idempotents if and only if it is a right bin.

**Keywords:** right *pp*-semigroups; right zero bands; strong spined products.

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## 1 Introduction

A right *pp*-semigroup, in brevity *rpp*-semigroup, is a semigroup  $S$  all whose principal right ideals  $aS^1$  ( $a \in S$ ), regarded as right  $S^1$ -systems, are projective. The class of *rpp*-semigroups and some of its special subclasses have been studied by J. B. Fountain [1], [2], [3], Y. Q. Guo, X.J. Guo, P.Y. Zhu and the authors (see [4]–[7] and [13]). Among the subclasses of *rpp*-semigroups, the class of *C-rpp*-semigroups is one of most important subclasses because it is a natural generalization of the Clifford semigroups. Namely, a Clifford semigroup is a strong semilattice of groups while a *C-rpp*-semigroup is a strong semilattice of left cancellative monoids [1]. By a *C-rpp*-semigroup  $S$ , we mean a *rpp*-semigroup  $S$  all whose idempotents are central, that is, the set  $E(S)$  of all idempotents of  $S$  lies in the center of  $S$ .

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In this paper, we study the structure of *rpp*-semigroups with left central idempotents. An idempotent  $e$  of a semigroup  $S$  is called a *left central idempotent* if  $xy = exy$  for all  $x, y \in S^1$ , with  $y$  different from the possible identity of  $S$ . It is clear that a *C-rpp*-semigroup is always a *rpp*-semigroup with left central idempotents but not conversely.

In generalizing the *C-rpp*-semigroups, we have recently studied the abundant semigroups with left central idempotents and also the perfect *C-rpp*-semigroups (see [13] and [7]). We have shown that an abundant semigroup with left central idempotents can be expressed by a strong semilattice of cancellative right stripes which are direct products of cancellative monoids and right zero bands (see [13]). By using similar arguments, one can prove that a *rpp*-semigroup with left central idempotents is a strong semilattice of left cancellative right stripes. By a left cancellative right stripe, we mean the direct product of a left cancellative monoid and a right zero band. On the other hand, it was shown in [7] that a perfect *C-rpp*-semigroup is a strong semilattice of left cancellative planks which are direct products of a left cancellative monoid and a rectangular band. In this way, one can easily visualize that the *rpp*-semigroups have a finer structure than the *C-rpp*-semigroups.

It has been already noticed by J.B. Fountain [2] that the  $\mathcal{L}^*$ -relation on a semigroup  $S$  is defined by:  $(a, b) \in \mathcal{L}^*$  if and only if  $a, b \in S$  are related by the Green's relation  $\mathcal{L}$  in some oversemigroup of  $S$ . He has shown in [2] that a semigroup  $S$  is *rpp* if for  $a \in S$ , there is an element  $e \in E(S)$  such that  $a\mathcal{L}^*e$ . Thus, if  $a, b$  are both regular elements of  $S$ , then  $a\mathcal{L}^*b$  if and only if  $a\mathcal{L}b$ . This result is a crucial result in studying the structure of *rpp*-semigroups with left central idempotents.

Recently, it has been shown in [7] that a perfect *C-rpp*-semigroup admits another structure, namely, it is the spined product of a *C-rpp*-semigroup and a normal band. We call this spined product a bin. It is natural to ask whether the *rpp*-semigroup with left central idempotents, as a generalization of *C-rpp*-semigroup, can be visualized by some other object which is similar to the bin. In this paper, we show that such kind of semigroups can be described by right bins, that is, the strong spined product of a *C-rpp*-semigroup and a right normal band. With the above terminology, we can say that the strong semilattice of left cancellative right stripes forms a right bin.

For other terminologies and notations not mentioned in this paper, the reader is referred to J.M. Howie [8], J.B. Fountain [2] and K.-P. Shum and Y.Q. Guo [12].

## 2 Left cancellative right stripes

In this paper, the concept of left cancellative right stripes is introduced and used to describe the structure of the  $rpp$ -semigroups with left central idempotents. We need the following definitions and lemmas.

**Definition 21.** A semigroup  $S$  is called a (*left*) *cancellative right stripe* if  $S$  is a direct product of a (*left*) cancellative monoid and a right zero band.

**Definition 22.** An idempotent  $e$  of a semigroup  $S$  is said to be a *left central idempotent* of  $S$  if for all  $x, y \in S^1$  with  $y$  different from the possible identity of  $S$ , we have  $xe y = exy$  in  $S$ .

The following lemmas describe the basic properties of a semigroup  $S$  which were given by J.B. Fountain in [2].

**Lemma 23** (see [2]). *Let  $a, b$  be elements of a semigroup  $S$ . Then the following conditions are equivalent:*

- (i)  $(a, b) \in \mathcal{L}^*$ ;
- (ii) For any  $x, y \in S^1$ ,  $ax = ay$  if and only if  $bx = by$ . ■

**Lemma 24** (see [2]). *If  $e$  is an idempotent of a semigroup  $S$ , then the following conditions are equivalent for  $a \in S$ :*

- (i)  $(a, e) \in \mathcal{L}^*$ ;
- (ii)  $a = ae$  and for any  $x, y \in S^1$ ,  $ax = ay \Rightarrow ex = ey$ . ■

For  $rpp$ -semigroups with left central idempotents, we have the following additional properties.

**Lemma 25.** *If  $S$  is a  $rpp$ -semigroup with left central idempotents, then every  $\mathcal{L}^*$ -class of  $S$  contains a unique idempotent.*

**Proof.** We recall that every  $\mathcal{L}^*$ -class of  $S$  contains an idempotent. Let  $e, f \in E(S)$  such that  $e\mathcal{L}^*f$ . Then, since  $e, f$  are regular elements, we have  $e\mathcal{L}f$ . Hence, by the definition of the Green's  $\mathcal{L}$ -relation, we have  $ef = e$  and  $fe = f$ . Since  $e$  is a left central idempotent, we have  $e = ef = e f f = f e f = f e = f$ . ■

Hereafter, by Lemma 2.5, we denote the  $\mathcal{L}^*$ -class of  $S$  containing an element  $a$  by  $L_a^*$  and its unique idempotent by  $a^*$ .

**Lemma 26.** *Let  $S$  be a rpp-semigroups with left central idempotents. Then  $\mathcal{L}^*$  is a congruence on  $S$ .*

**Proof.** Let  $a\mathcal{L}^*b$  for some  $a, b \in S$ . In order to show that  $\mathcal{L}^*$  is a left congruence, we need to show that  $ca\mathcal{L}^*cb$ , for any  $c \in S$ . For this purpose, we suppose that  $cax = cay$  for any  $x, y \in S^1$ . Then, by Lemma 2.4, we have  $c^*ax = c^*ay$ . Since  $a\mathcal{L}^*a^*$  and so by Lemma 2.4, we have  $aa^* = a$ , and thereby,  $c^*aa^*x = c^*aa^*y$ . Because  $S$  is a semigroup with left central idempotents, we have  $ac^*a^*x = ac^*a^*y$ . By  $a\mathcal{L}^*b$  and lemma 2.3, we also have  $bc^*a^*x = bc^*a^*y$ . Now, since  $c^*$  is a left central idempotent, we have  $c^*ba^*x = c^*ba^*y$ . By Lemma 2.3 again, we have  $cba^*x = cba^*y$ . Obviously,  $a^* = b^*$  by Lemma 2.5 and so we have  $cbx = cby$ . Conversely, if  $cbx = cby$ , then we can prove similarly that  $cax = cay$ . In other words, we have proved that  $ca\mathcal{L}^*cb$ . Clearly,  $\mathcal{L}^*$  is a right congruence on  $S$ . Hence  $\mathcal{L}^*$  is a congruence on  $S$ . ■

**Lemma 27.** *Let  $S$  be a rpp-semigroup with left central idempotents. Then  $(ab)^* = a^*b^*$  for all  $a, b \in S$ .*

**Proof.** Since  $a\mathcal{L}^*a^*, b\mathcal{L}^*b^*$  and  $\mathcal{L}^*$  is a congruence on  $S$ , we have  $ab\mathcal{L}^*a^*b^*$  for every  $a, b \in S$ . Thus, by Lemma 2.5, we have  $(ab)^* = a^*b^*$ . ■

**Lemma 28.** *Let  $S$  be a rpp-semigroup with left central idempotents. Then  $E(S)$  forms a right normal band, that is,  $efg = feg$  for any  $e, f, g \in E(S)$ .*

**Proof.** The proof is straightforward and we omit the proof. ■

We now define a binary relation  $\sigma$  on a rpp-semigroup  $S$  with left central idempotents by

$$a\sigma b \Leftrightarrow a^*b^* = b^* \text{ and } b^*a^* = a^* \text{ for all } a, b \in S.$$

**Lemma 29.** *If  $S$  is a rpp-semigroup with left central idempotents, then the relation  $\sigma$  defined above is a semilattice congruence on  $S$ .*

**Proof.** It is easy to check that  $\sigma$  is an equivalence relation on  $S$ . Let  $a, b, c \in S$  and assume  $a\sigma b$ . By Lemma 2.7, by definition of  $\sigma$  and  $E(S)$  being a right normal band, we have  $(ac)^*(bc)^* = a^*c^*b^*c^* = a^*b^*c^* = b^*c^* = (bc)^*$ , and analogously  $(bc)^*(ac)^* = (ac)^*$ . Hence  $ac\sigma bc$ . Thus,  $\sigma$  is a congruence on  $S$ . Also, by the definition of  $\sigma$ , we can see that for any  $a \in S$ ,  $a\sigma a^*$

and so  $a^2\sigma a$ . By Lemma 2.7 again, we can easily verify that  $ab\sigma ba$  for any  $a, b \in S$ . This shows that  $\sigma$  is indeed a semilattice congruence on  $S$ . ■

By using the above lemmas, we are now able to describe the  $rpp$ -semigroups with left central idempotents by using left cancellative right stripes. The following theorem is established.

**Theorem 210.** *Let  $S$  be a semigroup. Then the following conditions are equivalent:*

- (i)  $S$  is a  $rpp$ -semigroup with left central idempotents;
- (ii)  $S$  is a semilattice of left cancellative right stripes and  $E(S)$  is a right normal band;
- (iii)  $S$  is a strong semiattice of left cancellative right stripes.

**Proof.** (i)  $\Rightarrow$  (ii): Let  $S$  be a  $rpp$ -semigroup with left central idempotents. Then, by Lemma 2.9, there exists a semilattice congruence  $\sigma$  on  $S$  such that  $S = \bigcup_{\alpha \in Y} S_\alpha$  where  $Y = S/\sigma$  and each  $S_\alpha$  is a  $\sigma$ -class of  $S$ . We now show that  $S_\alpha$  can be expressed as a direct product of a left cancellative monoid and a right zero band. Let  $\Lambda_\alpha = S_\alpha \cap E(S)$ . Obviously  $\Lambda_\alpha$  is not empty and for every  $e, f \in \Lambda_\alpha$  we have  $ef = f$  and  $fe = e$ . Hence  $\Lambda_\alpha$  is a right zero band. Thus, for each  $\alpha \in Y$ , take a fixed  $e_\alpha \in \Lambda_\alpha$  and form  $M_\alpha = S_\alpha e_\alpha$ . Clearly,  $M_\alpha$  is a monoid. Suppose  $ab = ac$  for some  $a, b, c \in M_\alpha = S_\alpha e_\alpha$ . Then, by  $a\mathcal{L}^*a^*$  and by Lemma 2.4, we have  $a^*b = a^*c$  and so  $a^*be_\alpha = a^*ce_\alpha$ . Because  $a^*$  is left central, we have  $ba^*e_\alpha = ca^*e_\alpha$ . By  $a^*\sigma e_\alpha$ , we obtain that  $b = c$ . Thus,  $M_\alpha$  is a left cancellative monoid.

Now, we form the stripe  $M_\alpha \times \Lambda_\alpha$ . Define  $\varphi : M_\alpha \times \Lambda_\alpha \rightarrow S_\alpha$  by  $\varphi(x, f) = xf$  for  $x \in M_\alpha, f \in \Lambda_\alpha$ . Pick any  $(y, g) \in M_\alpha \times \Lambda_\alpha$ . Then, since the idempotents of  $S_\alpha$  are left central, we have  $\varphi(x, f)\varphi(y, g) = xfyg = xyfg = xyg = \varphi[(x, f)(y, g)]$ . This shows that  $\varphi$  is a homomorphism.

We claim that  $S_\alpha \cong M_\alpha \times \Lambda_\alpha$ . We first suppose that  $\varphi(x, f) = \varphi(y, g)$ , for  $(x, f), (y, g) \in M_\alpha \times \Lambda_\alpha$ . Then we have  $xf = yg$  and so  $xfe_\alpha = yge_\alpha$ . Since  $\Lambda_\alpha$  is a right zero band, we have  $xe_\alpha = ye_\alpha$ . Hence  $x = y$ , since  $x, y \in M_\alpha$ . Therefore  $xf = yg$ . Since  $x\sigma x^*$  and by Lemma 2.5, it leads to  $x^*f = x^*g$  and so  $f = g$  because  $\Lambda_\alpha$  is a right zero band. Hence  $(x, f) = (y, g)$ , and this shows that  $\varphi$  is injective. To see that  $\varphi$  is surjective, we just take any  $a \in S_\alpha$ . Then we have  $\varphi(ae_\alpha, a^*) = ae_\alpha a^* = a$ . Thus  $\varphi$  is surjective. Hence  $S_\alpha \cong M_\alpha \times \Lambda_\alpha$ . This shows that  $S$  is a semilattice of left cancellative right stripes.

Moreover, we can easily see that  $E(S)$  is a right normal band by Lemma 2.8.

(ii)  $\Rightarrow$  (iii): Suppose that  $S$  is a semilattice of left cancellative right stripes, that is,  $S = \bigcup_{\alpha \in Y} (M_\alpha \times \Lambda_\alpha)$ , where  $M_\alpha$  is a left cancellative monoid and  $\Lambda_\alpha$  is a right zero band. We have to show that there exists a structure homomorphism from  $M_\alpha \times \Lambda_\alpha$  to  $M_\beta \times \Lambda_\beta$  for every  $\alpha, \beta \in Y$  with  $\alpha \geq \beta$ . Let  $\alpha, \beta \in Y$  with  $\alpha \geq \beta$ . Then, for any  $a \in S_\alpha$ , pick a fixed  $e_\beta \in E(S_\beta)$ . Since  $S$  is a semilattice of  $S_\alpha$ , we have  $e_\beta a \in S_\beta$ . Thus, we can prove that the application  $\theta_{\alpha, \beta} : S_\alpha \rightarrow S_\beta$  defined by  $a\theta_{\alpha, \beta} = e_\beta a$  for every  $a \in S_\alpha$  is a homomorphism. Let  $a, b \in S_\alpha$ . Since  $e_\beta a \in S_\beta = M_\beta \times \Lambda_\beta$ , we can write  $e_\beta a = (u, i)$ , where  $u \in M_\beta, i \in \Lambda_\beta$ . Also, let  $g^2 = g = (1_\beta, i) \in M_\beta \times \Lambda_\beta$ , where  $1_\beta$  is the identity of the monoid  $M_\beta$ . Then, it is obvious that  $(e_\beta a)g = e_\beta a$  and  $e_\beta g = g$ . Putting  $b = (v, j)$  and  $h = (1_\alpha, j) \in M_\alpha \times \Lambda_\alpha$ , we get  $hb = b$ . Consequently, by the right normality of  $E(S)$ , we have  $e_\beta a e_\beta b = e_\beta a g e_\beta h b = e_\beta a g h b = e_\beta a b$ . This shows that  $a\theta_{\alpha, \beta} b\theta_{\alpha, \beta} = (ab)\theta_{\alpha, \beta}$  and so  $\theta_{\alpha, \beta}$  is a homomorphism. It is also trivial to see that  $\theta_{\alpha, \alpha}$  is an identity mapping. Moreover, for any  $\alpha, \beta, \gamma$  in  $Y$  with  $\alpha \geq \beta \geq \gamma$ , we pick any  $a = (u, i) \in S_\alpha = M_\alpha \times \Lambda_\alpha, h = (1_\alpha, i) \in M_\alpha \times \Lambda_\alpha$ . Then, obviously,  $ha = a$ . By the right normality of  $E(S)$  again, we have  $e_\gamma e_\beta h = e_\beta e_\gamma h = e_\beta e_\gamma e_\gamma h = e_\gamma h$ . This leads to  $a\theta_{\alpha, \beta} \theta_{\beta, \gamma} = e_\gamma (e_\beta a) = e_\gamma e_\beta h a = e_\gamma h a = e_\gamma a = a\theta_{\alpha, \gamma}$ , that is,  $\theta_{\alpha, \beta} \theta_{\beta, \gamma} = \theta_{\alpha, \gamma}$ .

Finally, we have to verify that for any  $\alpha, \beta \in Y$  with  $a \in S_\alpha, b \in S_\beta$ , we have  $ab = a\theta_{\alpha, \alpha\beta} b\theta_{\beta, \alpha\beta}$ . Since  $ab = e_{\alpha\beta}(ab)$ , we only need to show that  $e_{\alpha\beta} ab = e_{\alpha\beta} a e_{\alpha\beta} b$ . Since  $e_{\alpha\beta} a \in S_{\alpha\beta}$ , by using similar arguments as above, we can show that there exists  $f^2 = f \in S_{\alpha\beta}$  such that  $e_{\alpha\beta} a f = e_{\alpha\beta} a$ . Likewise, for any  $b \in S_\beta$ , there exists  $e_\beta^2 = e_\beta \in S_\beta$  such that  $e_\beta b = b$ . Thus, by the right normality of  $E(S)$ , we have

$$\begin{aligned} e_{\alpha\beta} a e_{\alpha\beta} b &= (e_{\alpha\beta} a f) e_{\alpha\beta} b = e_{\alpha\beta} a f e_{\alpha\beta} e_\beta b \\ &= e_{\alpha\beta} a e_{\alpha\beta} f e_\beta b = e_{\alpha\beta} a f e_\beta b \\ &= (e_{\alpha\beta} a f) (e_\beta b) = e_{\alpha\beta} a b \end{aligned}$$

This shows that  $ab = a\theta_{\alpha, \alpha\beta} b\theta_{\beta, \alpha\beta}$ . Hence,  $S_\alpha$  is a strong semilattice of  $M_\alpha \times \Lambda_\alpha$ .

(iii)  $\Rightarrow$  (i): Suppose that  $S = [Y; S_\alpha, \theta_{\alpha, \beta}]$  is a strong semilattice of the right stripes  $S_\alpha = M_\alpha \times \Lambda_\alpha$ , where  $M_\alpha$  is a left cancellative monoid and  $\Lambda_\alpha$  is a right zero band. We first prove that every  $e^2 = e \in E(S)$  is left central. For this purpose, we let  $x, y \in S^1$  with  $y \neq 1$ . Then, there exist  $\alpha, \beta, \gamma \in Y$  such that  $x \in S_\alpha^1, y \in S_\beta^1$  and  $e \in S_\gamma$ . Write  $\delta = \alpha\beta\gamma$ ,  $x\theta_{\alpha, \delta} = (u, i) \in M_\delta \times \Lambda_\delta, y\theta_{\beta, \delta} = (v, j) \in M_\delta \times \Lambda_\delta$  and  $e\theta_{\gamma, \delta} = (1_\delta, k)$ . Then,

we have  $xey = x\theta_{\alpha,\delta}e\theta_{\gamma,\delta}y\theta_{\beta,\delta} = (u, i)(1_\delta, k)(v, j) = (uv, j)$ . Similarly, we can prove that  $xey = (1_\delta, k)(u, i)(v, j) = (uv, j)$ . Consequently,  $xey = exy$  so that every idempotent of  $S$  is left central.

To show that (i) holds, we still need to show that  $S$  is a  $rpp$ -semigroup. Obviously, for any  $a \in S$ , there exists  $\alpha \in Y$  such that  $a = (u, i) \in S_\alpha = M_\alpha \times \Lambda_\alpha$ . Now, let  $f = (1_\alpha, i)$ . Then, it is clear that  $f^2 = f$  and  $af = a$ . Also, for any  $x, y \in S^1$  (with  $y \neq 1$ ), we assume that  $ax = ay$ . Then, by the left cancellativity of  $M_\alpha$  and that  $S$  is a strong semilattice of  $M_\alpha \times \Lambda_\alpha$ , we can verify that  $fx = fy$ . By Lemma 2.4, we have  $a\mathcal{L}^*f$ . In other words, this shows that every  $\mathcal{L}^*$ -class of  $S$  contains an idempotent. Thus, by definition,  $S$  is a  $rpp$ -semigroup. ■

### 3 Main structure theorem

In this section, we give another structure for the  $rpp$ -semigroups with left central idempotents. We will describe such kind of semigroups in terms of right bins and hence generalize a corresponding result in abundant semigroups with left central idempotents. We first cite the following results which are useful in proving our main theorem.

**Lemma 31** (cf. [1] and [2]). (i) *A semigroup  $M$  is a  $C$ - $rpp$ -semigroup if and only if  $M$  is a strong semilattice  $Y$  of left cancellative monoids  $M_\alpha$ , that is,  $M = [Y; M_\alpha, \varphi_{\alpha,\beta}]$ . (ii) *A semigroup  $M$  is a  $C$ - $a$ -semigroup (that is, an abundant semigroup in which all idempotents of  $S$  are central) if and only if  $M$  is a strong semilattice of cancellative monoids.* ■*

**Lemma 32** (see [8]). *A semigroup  $\Lambda$  is a right normal band if and only if  $\Lambda$  is a strong semilattice  $Y$  of right zero bands  $\Lambda_\alpha$ , that is,  $\Lambda = [Y; \Lambda_\alpha, \theta_{\alpha,\beta}]$ .* ■

Now, let  $T = [Y; T_\alpha, \varphi_{\alpha,\beta}]$  and  $I = [Y; I_\alpha, \theta_{\alpha,\beta}]$  be two strong semilattices of semigroups. For each  $\alpha \in Y$ , we form the direct product  $S_\alpha = T_\alpha \times I_\alpha$ . Write  $S = \cup_{\alpha \in Y} S_\alpha = \cup_{\alpha \in Y} (T_\alpha \times I_\alpha)$  and define a binary operation “ $*$ ” on  $S$  by

$$(t_1, i_1) * (t_2, i_2) = (t_1\varphi_{\alpha,\alpha\beta} t_2\varphi_{\beta,\alpha\beta}, i_1\theta_{\alpha,\alpha\beta} i_2\theta_{\beta,\alpha\beta})$$

for any  $(t_1, i_1) \in S_\alpha, (t_2, i_2) \in S_\beta$ . Then, it can be easily checked that  $(S, *)$  is a semigroup. We then call  $S$  the *strong spined product of  $T$  and  $I$* . Denote this spined product by  $S = T \times I$ .

The so-defined semigroup  $S$  is a *spined product of  $T$  and  $I$*  according to the definition given by N. Kimura in [9]. In fact it is enough to take  $H = Y$  and  $\phi$  and  $\psi$  as the natural homomorphisms of  $T$  and  $I$  on the semilattice  $Y$ . We remark here that we have also used the concept of “spined product” of semigroups to study the structure of quasi-rectangular groups in [11].

In view of Lemmas 3.1 and 3.2, we introduce the following concepts of bins and casks.

**Definition 33.**

- (i) The strong spined product of a  $C$ -*rpp*-semigroup  $M$  and a right (left) normal band  $\Lambda$  is called a *right (left) bin*.
- (ii) The strong spined product of a  $C$ -*a*-semigroup and a right (left) normal band is called a *right (left) cask*.
- (iii) The strong spined product of a Clifford semigroup and a right (left) normal band is called a *right (left) barrier*.

We now formulate another structure theorem for *rpp*-semigroups with left central idempotents as follows:

**Theorem 34.** *A semigroup  $S$  is a  $rpp$ -semigroup with left central idempotents if and only if it is a right bin.*

**Proof.**  $\Leftarrow$ ) By the definition of right bin, it means that  $S$  is a strong spined product of a  $C$ -*rpp*-semigroup  $M$  and a right normal band  $\Lambda$ , that is,  $S = M \underline{\times} \Lambda$ . By the results in [1] and [8], we have  $M = [Y; M_\alpha, \varphi_{\alpha,\beta}]$ ,  $\Lambda = [Y; \Lambda_\alpha, \theta_{\alpha,\beta}]$  and  $S = \bigcup_{\alpha \in Y} S_\alpha$ , where each  $M_\alpha$  is a left cancellative monoid,  $\Lambda_\alpha$  is a right zero band and  $S_\alpha = M_\alpha \times \Lambda_\alpha$ . Now, we can easily see that  $E = \{(1_\alpha, i) \mid 1_\alpha \text{ is the identity of } M_\alpha \text{ and } i \in \Lambda_\alpha\}$  is the set of idempotents of  $S$ . To see that  $S$  is a *rpp*-semigroup, we just pick any  $a \in S = \bigcup_{\alpha \in Y} S_\alpha$ . Hence  $a = (u, i) \in S_\alpha$  for some  $\alpha \in Y$  and  $e = (1_\alpha, i)$  is an idempotent in  $S_\alpha$ . Let  $x = (s, m) \in S_\beta^1$  and  $y = (t, n) \in S_\gamma^1$ , for some  $\beta, \gamma \in Y$  and assume  $ax = ay$ , that is,  $(u, i)(s, m) = (u, i)(t, n)$ . Then, by the multiplication “\*” on  $S$  and by a result of J.M. Howie ([8], Corollary 5.16, Chapter IV), we have  $u\varphi_{\alpha,\alpha\beta}s\varphi_{\beta,\alpha\beta} = u\varphi_{\alpha,\alpha\gamma}t\varphi_{\gamma,\alpha\gamma}$ ,  $m\theta_{\beta,\alpha\beta} = i\theta_{\alpha,\alpha\beta}m\theta_{\beta,\alpha\beta} = i\theta_{\alpha,\alpha\beta}n\theta_{\gamma,\alpha\gamma} = n\theta_{\gamma,\alpha\gamma}$  and  $\alpha\beta = \alpha\gamma$ . Since the monoid  $M_{\alpha\beta}$  is left cancellative, we have  $s\varphi_{\beta,\alpha\beta} = t\varphi_{\gamma,\alpha\gamma}$ . Thereby, we can easily verify that  $(1_\alpha, i)(s, m) = (1_\alpha, i)(t, n)$ . This shows that  $a\mathcal{L}^*e$ . Thus  $S$  is a *rpp*-semigroup.



Now it is easy to verify that  $aeb = eab$  holds for all  $a, b \in S$  ( $b \neq 1$ ) and any  $e \in E(S)$ . Hence we have shown that  $S = M \times \Lambda$  is a  $rpp$ -semigroup with left central idempotents.

$\implies$ ) Let  $S$  be a  $rpp$ -semigroup with left central idempotents. Then, by Theorem 2.10,  $S$  is a strong semilattice of left cancellative right stripes, that is,  $S$  is a strong semilattice of  $M_\alpha \times \Lambda_\alpha$ , where  $M_\alpha$  is a left cancellative monoid and  $\Lambda_\alpha$  is a right zero band, for all  $\alpha \in Y$ .

We now form  $M = \cup_{\alpha \in Y} M_\alpha$  and  $\Lambda = \cup_{\alpha \in Y} \Lambda_\alpha$ . We first claim that  $M$  is a strong semilattice of  $M_\alpha$ . For this purpose, we pick a fixed element  $i_\alpha \in \Lambda_\alpha$  and denote by  $1_\alpha$  the identity of the monoid  $M_\alpha$ . Then, by Theorem 2.10, for any  $\alpha, \beta \in Y$  with  $\alpha \geq \beta$  and for  $(u, i) \in M_\alpha \times \Lambda_\alpha$ , we have  $(1_\beta, i_\beta)(u, i) = (u', -) \in M_\beta \times \Lambda_\beta$ , where  $u'$  is independent by the choice of  $i$ .

In fact, for any  $j \in \Lambda_\alpha$ , we have

$$\begin{aligned} (1_\beta, i_\beta)(u, j) &= (1_\beta, i_\beta)(u, i)(1_\alpha, j) \\ &= (u', -)(1_\alpha, j) = (u', -). \end{aligned}$$

Then, we can define a mapping from  $M_\alpha$  into  $M_\beta$  by  $\varphi_{\alpha, \beta} : u \mapsto u\varphi_{\alpha, \beta}$  satisfying

$$(1_\beta, i_\beta)(u, i) = (u\varphi_{\alpha, \beta}, -).$$

Let  $\alpha, \beta \in Y$  with  $\alpha \geq \beta$ , Then, for every  $(u, i), (v, j) \in M_\alpha \times \Lambda_\alpha$ , we have

$$\begin{aligned} (1_\beta, i_\beta)(uv, j) &= (1_\beta, i_\beta)[(u, i)(v, j)] \\ &= [(1_\beta, i_\beta)(u, i)][(1_\beta, i_\beta)(v, j)], \end{aligned}$$

because the idempotents of  $S$  are left central. Hence,  $(uv)\varphi_{\alpha, \beta} = u\varphi_{\alpha, \beta} v\varphi_{\alpha, \beta}$  for any  $u, v \in M_\alpha$ . Trivially,  $\varphi_{\alpha, \alpha}$  is the identity mapping on  $M_\alpha$ . Now let  $\alpha, \beta, \gamma \in Y$  with  $\alpha \geq \beta \geq \gamma$ , and let  $(u, i) \in M_\alpha \times \Lambda_\alpha$ . Then it follows that

$$\begin{aligned} (1_\gamma, i_\gamma)[(1_\beta, i_\beta)(u, i)] &= [(1_\gamma, i_\gamma)(1_\beta, i_\beta)(1_\gamma, i_\gamma)](u, i) \\ &= (1_\gamma, i_\gamma)(u, i). \end{aligned}$$

Thus,  $\varphi_{\alpha, \beta}$  is a structure homomorphism. Hence, the set  $M = \cup_{\alpha \in Y} M_\alpha$  with binary operation “ $*$ ” defined by

$$u * v = u\varphi_{\alpha, \alpha\beta} v\varphi_{\beta, \alpha\beta} \text{ for any } u \in M_\alpha \text{ and } v \in M_\beta,$$

forms a strong semilattice of  $M_\alpha$ , denoted by  $M = [Y; M_\alpha, \varphi_{\alpha, \beta}]$ . By Lemma 3.1 (i),  $M$  is a  $C$ - $rpp$ -semigroup.

We now consider  $\Lambda = \cup_{\alpha \in Y} \Lambda_\alpha$ , where each  $\Lambda_\alpha$  is a right zero band. It can be easily observed that the mapping  $\eta : E(S) \longrightarrow \Lambda$  defined by  $(1_\alpha, i)\eta = i$  is a bijective. Moreover if we define a multiplication “ $\circ$ ” on  $\Lambda$ , putting for every  $i \in \Lambda_\alpha, j \in \Lambda_\beta, i \circ j = k$  if and only if  $(1_\alpha, i)(1_\beta, j) = (1_{\alpha\beta}, k)$ . It can be easily seen that  $E(S)$  is isomorphic to  $\Lambda$  under  $\eta$ .

For any  $\alpha, \beta \in Y$  with  $\alpha \geq \beta$ , we have  $(1_\beta, i_\beta)(1_\alpha, i) = (1_\beta, k^*) \in E_\beta$  for  $(1_\alpha, i) \in E_\alpha$ . Thus, we can define a mapping  $\theta_{\alpha, \beta} : i \longmapsto i\theta_{\alpha, \beta}$  from  $\Lambda_\alpha$  into  $\Lambda_\beta$  by  $(1_\beta, i_\beta)(1_\alpha, i) = (1_\beta, i\theta_{\alpha, \beta})$ . Obviously,  $\theta_{\alpha, \alpha}$  is the identity map on  $\Lambda_\alpha$  and  $\theta_{\alpha, \beta}$  is a semigroup homomorphism. It is routine to check that for any  $\alpha, \beta, \gamma \in Y$  with  $\alpha \geq \beta \geq \gamma$ , we have  $\theta_{\alpha, \beta}\theta_{\beta, \gamma} = \theta_{\alpha, \gamma}$ . Hence,  $\theta_{\alpha, \beta}$  is a structure homomorphism. In other words, if we define  $i \circ j = j\theta_{\beta, \alpha\beta}$  for any  $i \in \Lambda_\alpha, j \in \Lambda_\beta$  then the semigroup  $(\Lambda, \circ)$  forms a strong semilattice of right zero bands, namely  $\Lambda = [Y; \Lambda_\alpha, \theta_{\alpha, \beta}]$ . By Lemma 3.2, we know that  $\Lambda$  is a right normal band.

Finally, we verify that the multiplication on  $S$  coincides with the semigroup multiplication of the strong spined product of the  $C$ -rpp-semigroup  $M$  and the right normal band  $\Lambda$ . In fact, by picking  $(u, i) \in M_\alpha \times \Lambda_\alpha$  and  $(v, j) \in M_\beta \times \Lambda_\beta$ , we have

$$\begin{aligned} (u, i)(v, j) &= (u, i)(1_\alpha, i)(1_\beta, j)(v, j) \\ &= (u, i)[(1_{\alpha\beta}, i_{\alpha\beta})(1_\alpha, i)(1_\beta, j)](v, j) \\ &= [(1_{\alpha\beta}, i_{\alpha\beta})(u, i)][(1_{\alpha\beta}, i_{\alpha\beta})(v, j)](1_{\alpha\beta}, i_{\alpha\beta})(1_\beta, j) \\ &= (u\varphi_{\alpha, \alpha\beta}, -)(v\varphi_{\beta, \alpha\beta}, -)(1_{\alpha\beta}, j\theta_{\beta, \alpha\beta}) \\ &= (u\varphi_{\alpha, \alpha\beta} \ v\varphi_{\beta, \alpha\beta}, j\theta_{\beta, \alpha\beta}) = (u * v, i \circ j). \end{aligned}$$

Thus, we have shown that  $S \cong M \underline{\times} I$  and the proof is completed.  $\blacksquare$

The following results are some special cases of Theorem 3.4.

**Corollary 35.**

- (i) *Let  $M$  be a  $C$ -a-semigroup and  $\Lambda$  a right normal band. Then the strong spined product of  $M \underline{\times} \Lambda$  is an abundant semigroup with left central idempotents. Conversely, every abundant semigroup  $S$  with left central idempotents can be expressed by the strong spined product  $M \underline{\times} \Lambda$ .*
- (ii) *A semigroup  $S$  is a regular semigroup with left central idempotents if and only if  $S$  is a right barrier.*  $\blacksquare$

**Note:** Since a left  $C$ - $a$ -semigroup is a strong semilattice of cancellative monoids, by the terminology of cask, we can therefore state that a semigroup  $S$  is an abundant semigroup with left central idempotents if and only if it is a right cask.

## 4 Some additional properties

In this section, we give some further properties for the  $rpp$ -semigroups with left central idempotents.

**Theorem 41.** *A  $rpp$ -semigroup with left central idempotents is a right normal band of left cancellative monoids.*

**Proof.** Let  $S$  be a  $rpp$ -semigroup with left central idempotents. Then each  $\mathcal{L}^*$ -class of  $S$  contains a unique idempotent by Lemma 2.5. Also, by Lemma 2.6, we know that  $\mathcal{L}^*$  is a congruence on  $S$ . By Lemma 2.7, we know that  $(ab)^* = a^*b^*$  for any  $a, b \in S$ . Now, we consider the quotient semigroup  $S/\mathcal{L}^*$ . Define  $\theta : S/\mathcal{L}^* \rightarrow E(S)$  by  $\theta : L_a^* \mapsto a^*$ . Then it is trivial to see that  $\theta$  is an isomorphism. By Theorem 2.10, the quotient semigroup  $S/\mathcal{L}^*$  is a right normal band. It remains to show that each  $L_a^*$  is a left cancellative monoid. Since  $S$  is a  $rpp$ -semigroup with left central idempotents, each  $L_a^*$  contains the unique idempotent  $a^*$  such that  $aa^* = a$  and so  $a^*a = a^*(aa^*) = a(a^*a^*) = aa^* = a$ . This shows that  $a^*$  is the identity element of the monoid  $L_a^*$ . Finally, according to Lemma 2.4, it is known that in the  $rpp$ -semigroup  $S$ ,  $a\mathcal{L}^*e$  if and only if  $ae = a$  and for all  $x, y \in S^1$ ,  $ax = ay \Rightarrow ex = ey$ . Thus, if we assume that  $ab = ac$  for  $a, b, c \in L_a^*$ , then  $a^*b = a^*c$ . Thereby, we have  $(a^*b)a^* = (a^*c)a^*$ . This leads to  $b(a^*a^*) = c(a^*a^*)$ . Since  $a^*$  is the identity element of the monoid  $L_a^*$ , we obtain immediately that  $b = c$ . This shows that  $L_a^*$  is indeed a left cancellative monoid and the proof is completed. ■

**Definition 42.** A relation  $\tau$  defined by  $(a, b) \in \tau$  on a semigroup  $S$  is called *E-diagonal* if and only if for  $a, b \in S$  there exists  $e \in E(S)$  such that  $ae = be$ .

**Theorem 43.** *Let  $S$  be a  $rpp$ -semigroup with left central idempotents. Then an E-diagonal relation  $\tau$  is the smallest left cancellative monoid congruence on  $S$ .*

**Proof.** Since  $S$  is a  $rpp$ -semigroup with left central idempotents, we have  $aa^* = a$ , where  $a^*$  is the unique idempotent of the  $\mathcal{L}^*$ -class containing  $a$ .

Let  $\tau$  be an  $E$ -diagonal relation on  $S$ . Clearly,  $\tau$  is reflexive and we can easily check that  $\tau$  is an equivalence relation on  $S$ . For  $(a, b) \in \tau$ , by definition, there exists  $e \in E(S)$  such that  $ae = be$ . Thus for any  $c \in S$ , we have  $acec^* = aecc^* = becc^* = bcec^*$ . This shows that  $(ac, bc) \in \tau$ , that is,  $\tau$  is a right congruence on  $S$ . Clearly,  $\tau$  is a left congruence on  $S$ . Thus,  $\tau$  is a congruence on  $S$  and hence  $S/\tau$  is a semigroup. To see that  $S/\tau$  is a left cancellative monoid, we let  $a\tau b\tau = a\tau c\tau$  for  $a, b, c \in S$ . Since  $(ab)\tau = (ac)\tau$ , there exists  $e \in E(S)$  such that  $abe = ace$ . Since  $S$  is rpp with left central idempotents, we have  $a^*be = a^*ce$ . This implies that  $ba^*e = ca^*e$ . Clearly,  $a^*e \in E(S)$  and consequently  $b\tau = c\tau$ . Since  $efg = feg$  for any  $e, f, g \in E(S)$ , we have  $efe = fee$ . This implies that  $(ef)\tau = (fe)\tau = (fee)\tau = (efe)\tau$ . Thereby,  $e\tau f\tau = e\tau f\tau e\tau$  and  $e\tau e\tau f\tau = e\tau f\tau$ . It then follows that  $f\tau = f\tau e\tau$  and  $f\tau = e\tau f\tau$  since  $S/\tau$  is left cancellative. Similarly, we have  $e\tau = e\tau f\tau$  and so  $e\tau = f\tau$  for any  $e, f \in E$ . Thus,  $e\tau$  is the identity element of the semigroup  $S/\tau$  for any  $e \in E(S)$ . This shows that  $S/\tau$  is a left cancellative monoid.

Finally, we show that  $\tau$  is the smallest left cancellative monoid congruence defined on  $S$ . In fact, if  $\rho$  is a left cancellative monoid congruence on  $S$  and  $(a, b) \in \tau$ , then by Definition 4.2, there exists  $e \in E(S)$  such that  $ae = be$ . Hence,  $a\tau e\tau = b\tau e\tau$  and  $a\tau = b\tau$ , because  $\rho$  is a cancellative congruence. This shows that  $\tau \subseteq \rho$  as required. ■

**Corollary 44.** *Let  $S$  be an abundant semigroup with left central idempotents. Then the  $E$ -diagonal relation  $\tau$ , defined by  $(a, b) \in \tau$  if and only if there exists  $e \in E(S)$  such that  $ae = be$ , is the smallest cancellative monoid congruence on  $S$ .* ■

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