

## MODIFICATIONS OF CSÁKÁNY'S THEOREM

IVAN CHAJDA

*Department of Algebra and Geometry, Palacký University of Olomouc*  
*Tomkova 40, Cz-779 00 Olomouc, Czech Republic*

**e-mail:** chajda@risc.upol.cz

### Abstract

Varieties whose algebras have no idempotent element were characterized by B. Csákány by the property that no proper subalgebra of an algebra of such a variety is a congruence class. We simplify this result for permutable varieties and we give a local version of the theorem for varieties with nullary operations.

**Keywords:** congruence class, idempotent element, permutable variety, Mal'cev condition.

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## 1 Introduction

B. Csákány [2] proved the following statement:

**Proposition.** *For a variety  $\mathcal{V}$ , the following conditions are equivalent:*

- (i) *None of algebras in  $\mathcal{V}$  having at least two elements have idempotent elements;*
- (ii) *No algebra  $\mathcal{A} \in \mathcal{V}$  has a proper subalgebra whose carrier is a class of some  $\theta \in \text{Con } \mathcal{A}$  ;*
- (iii) *There exist  $n \in \mathbb{N}$ , ternary terms  $p_1, \dots, p_n$ , and unary terms  $f_1, \dots, f_n, g_1, \dots, g_n$  such that the identities*

$$x = p(f_1(x), x, y),$$

$$p_i(g_i(x), x, y) = p_{i+1}(f_{i+1}(x), x, y), \quad \text{for } i = 1, \dots, n-1,$$

$$y = p_n(g_n(x), x, y)$$

*hold in  $\mathcal{V}$ .*

It was recognized by J. Kollár [3] that each of the equivalent conditions of the Proposition is equivalent to

- (iv) For all  $\mathcal{A} \in \mathcal{V}$ , the greatest congruence  $\iota_{\mathcal{A}}$  on  $\mathcal{A}$  is a compact element of  $\text{Con } \mathcal{A}$ .

Analyzing the proof of Proposition, we find out that these conditions are also equivalent to

- (v)  $\theta(F_{\mathcal{V}}(x) \times F_{\mathcal{V}}(x)) = F_{\mathcal{V}}(x, y) \times F_{\mathcal{V}}(x, y)$  in  $\text{Con } F_{\mathcal{V}}(x, y)$ ,

where  $F_{\mathcal{V}}(x_1, \dots, x_n)$  denotes the free algebra of  $\mathcal{V}$  generated by the set  $\{x_1, \dots, x_n\}$  of free generators, and  $\theta(M \times M)$  denotes the least congruence containing the set  $M \times M$ .

From this point of view, (v) can be modified by several ways. We can consider a variety  $\mathcal{V}$  with constants (i.e. nullary operations) and we can omit a free variable on the left-hand side of (v) to obtain

- (vi)  $\theta(F_{\mathcal{V}}(\emptyset) \times F_{\mathcal{V}}(\emptyset)) = F_{\mathcal{V}}(x, y) \times F_{\mathcal{V}}(x, y)$  in  $\text{Con } F_{\mathcal{V}}(x, y)$ .

This condition used in [1] it is equivalent to the property that  $\iota_{\mathcal{A}}$  is generated by the set of nullary operations for each  $\mathcal{A} \in \mathcal{V}$ .

Further, we can also

- (a) simplify Csákány's original result for permutable varieties;  
 (b) omit one free variable in both sides of (v) to obtain  
 (vii)  $\theta(F_{\mathcal{V}}(\emptyset) \times F_{\mathcal{V}}(\emptyset)) = F_{\mathcal{V}}(x) \times F_{\mathcal{V}}(x)$  in  $\text{Con } F_{\mathcal{V}}(x)$ .

In the second case we obtain a local version of Csákány's theorem. These modifications are treated in this paper.

## 2 Results

**Theorem 1.** *Let  $\mathcal{V}$  be a permutable variety. The following conditions are equivalent:*

- (i) *None of algebras in  $\mathcal{V}$  having at least two elements have idempotent elements;*  
 (ii) *No algebra  $\mathcal{A} \in \mathcal{V}$  has a proper subalgebra whose carrier is a class of some  $\theta \in \text{Con } \mathcal{A}$ ;*  
 (iii) *There exist  $n \in \mathbb{N}$  and a  $(2 + n)$ -ary term  $p$  and unary terms  $v_1, \dots, v_n, w_1, \dots, w_n$  such that the identities*

$$x = p(x, y, v_1(x), \dots, v_n(x)),$$

$$y = p(x, y, w_1(x), \dots, w_n(x))$$

hold in  $\mathcal{V}$ .

**Proof.** The equivalence of (i) and (ii) is proven by the Proposition. Prove (ii) $\Rightarrow$ (iii): Set  $\mathcal{A} = F_{\mathcal{V}}(x, y)$  and  $\mathcal{B} = F_{\mathcal{V}}(x)$ . Let  $\theta = \theta(B \times B) \in \text{Con } \mathcal{A}$  (where  $B$  is the carrier of  $\mathcal{B}$ ). Since  $\mathcal{B}$  is a subalgebra of  $\mathcal{A}$ , we have  $\theta(B \times B) = \theta = \iota_{\mathcal{A}}$  by (ii). However,  $\mathcal{V}$  is permutable; thus  $\theta(B \times B) = R(B \times B)$ , the least reflexive and compatible relation on  $\mathcal{A}$  containing the set  $B \times B$ . It follows  $\iota_{\mathcal{A}} = R(B \times B)$  which yields  $\langle x, y \rangle \in R(B \times B)$ . Hence, there exists a  $(2+n)$ -ary term  $p$  and elements  $b_1, \dots, b_n, b'_1, \dots, b'_n \in B$  such that

$$x = p(x, y, b_1, \dots, b_n) \text{ and } y = p(x, y, b'_1, \dots, b'_n).$$

Since  $b_i, b'_i \in F_{\mathcal{V}}(x)$ , there are unary terms  $v_i, w_i$  with  $b_i = v_i(x)$  and  $b'_i = w_i(x)$ ,  $i = 1, \dots, n$ .

For (iii) $\Rightarrow$ (i) let  $\mathcal{A} \in \mathcal{V}$  with  $|A| > 1$  and suppose that  $a \in A$  is an idempotent element. Let  $b \in A \setminus \{a\}$ . Then  $v_i(a) = a = w_i(a)$  and, by (iii),  $a = p(a, b, v_i(a), \dots, v_n(a)) = p(a, b, a, \dots, a) = p(a, b, w_1(a), \dots, w_n(a)) = b$ , a contradiction.  $\blacksquare$

**Example 1.** For a variety  $\mathcal{R}$  of rings with 1, we can set  $n = 2$ ,  $v_1(x) = 1 = w_2(x)$ ,  $v_2(x) = 0 = w_1(x)$  and  $p(x, y, a, b) = ax + by$ . Hence, it follows that the reduct of a ring with 1, determined by the terms  $0, 1, x - y + z$ , and  $xy$ , generates a permutable variety with no idempotent elements.

In this section we consider only varieties  $\mathcal{V}$  having a nullary operation which will be denoted by  $0$ ; it is usually called a *constant*. For  $\mathcal{A} \in \mathcal{V}$ , this constant will be denoted by  $0_{\mathcal{A}}$ . We do not restrict the number of nullary operations of  $\mathcal{V}$  but this  $0$  will be considered to be fixed.

**Theorem 2.** *Let  $\mathcal{V}$  be a variety with  $0$ . The following conditions are equivalent:*

- (i) *No  $\mathcal{A} \in \mathcal{V}$  having at least two elements has  $0_{\mathcal{A}}$  as an idempotent element;*
- (ii) *For each  $\mathcal{A} \in \mathcal{V}$  and each  $\theta \in \text{Con } \mathcal{A}$ ,  $[0_{\mathcal{A}}]_{\theta}$  is not a proper subalgebra of  $\mathcal{A}$ ;*
- (iii) *There exist  $n \in \mathbb{N}$ , binary terms  $q_1, \dots, q_n$ , and unary terms  $v_1, \dots, v_n, w_1, \dots, w_n$  such that the identities*

$$x = q_1(x, v_1(0)),$$

$$q_1(x, w_i(0)) = q_{i+1}(x, v_{i+1}(0)), \quad i = 1, \dots, n-1,$$

$$0 = q_n(x, w_n(0))$$

hold in  $\mathcal{V}$ .

**Proof.** (i) $\Rightarrow$ (ii): Let  $\mathcal{A} \in \mathcal{V}$ ,  $|A| > 1$ ,  $\theta \in \text{Con } \mathcal{A}$  and suppose  $[0_A]_\theta \neq A$ . Then  $\mathcal{A}/\theta$  has at least two elements, and, of course,  $0_{\mathcal{A}/\theta} = [0_A]_\theta$ . Since  $0_{\mathcal{A}/\theta}$  is not an idempotent of  $\mathcal{A}/\theta \in \mathcal{V}$ ,  $[0_A]_\theta$  cannot be a subalgebra of  $\mathcal{A}$ .

(ii) $\Rightarrow$ (iii): Set  $\mathcal{A} = \mathcal{F}_{\mathcal{V}}(x)$  and  $\mathcal{B} = \mathcal{F}_{\mathcal{V}}(\emptyset)$ . Let  $\theta = \theta(B \times B)$  in  $\text{Con } \mathcal{A}$ . Since  $0_A = 0_B \in B$ , the class  $[0_A]_\theta$  contains  $B$ . Hence, for every  $n$ -ary operation  $f$  of the type of  $\mathcal{V}$ , for every  $c_1, \dots, c_n \in [0_A]_\theta$ , and every  $b_1, \dots, b_n \in B$  we have  $\langle c_i, b_i \rangle \in \theta$  ( $i = 1, \dots, n$ ); thus, also  $\langle f(c_1, \dots, c_n), f(b_1, \dots, b_n) \rangle \in \theta$ . But  $f(b_1, \dots, b_n) \in B \subseteq [0_A]_\theta$ , whence  $f(c_1, \dots, c_n) \in [0_A]_\theta$ . Thus,  $[0_A]_\theta$  is a subalgebra of  $\mathcal{A}$ . In account of (ii),  $[0_A]_\theta = A$ ; thus,  $\langle x, 0 \rangle \in \theta(B \times B)$ . Hence, there exist  $d_0, d_1, \dots, d_n \in A$  such that  $d_0 = x$ ,  $d_n = 0$  and  $\langle d_{i-1}, d_i \rangle = \langle \varphi_i(b_i), \varphi_i(b'_i) \rangle$  ( $i = 1, \dots, n$ ) for some  $b_i, b'_i \in B$  and unary polynomials  $\varphi_i$  over  $\mathcal{A}$ . Thus  $b_i = v_i(0)$ ,  $b'_i = w_i(0)$  for some unary terms  $v_i, w_i$ . Of course,  $\varphi_i(z) = q_i(x, z)$  for some binary terms  $q_1, \dots, q_n$ . The condition (iii) is evident.

(iii) $\Rightarrow$ (i): Let  $\mathcal{A} \in \mathcal{V}$ ,  $|A| > 1$ ,  $0_A \neq a \in A$ . Suppose that  $0_A$  is an idempotent of  $\mathcal{A}$ . Then  $v_i(0_A) = 0_A = w_i(0_A)$  and  
 $a = q_1(a, v_1(0_A)) = q_1(a, 0_A) = q_1(a, w_1(0_A)) =$   
 $q_2(a, v_2(0_A)) = q_2(a, 0_A) = q_2(a, w_2(0_A)) = \dots = 0_A,$   
 a contradiction. ■

**Example 2.** For a variety  $\mathcal{P}$  of pseudocomplemented semilattices, we set  $n = 1$ ,  $v_1(x) = x^*$ ,  $w_1(x) = x$  and  $q_1(x, y) = x \wedge y$ . Then

$$q_1(x, v_1(0)) = x \wedge 0^* = x,$$

$$q_1(x, w_1(0)) = x \wedge 0 = 0.$$

Analogously as previously, we can simplify Theorem 2 for permutable varieties.

**Theorem 3.** Let  $\mathcal{V}$  be a permutable variety with 0. The following conditions are equivalent:

- (i) No  $\mathcal{A} \in \mathcal{V}$  consisting of at least two elements has  $0_A$  as an idempotent element;

- (ii) For each  $\mathcal{A} \in \mathcal{V}$  and each  $\theta \in \text{Con } \mathcal{A}$ ,  $[0_{\mathcal{A}}]_{\theta}$  is not a proper subalgebra of  $\mathcal{A}$ ;
- (iii) There exist  $n \in \mathbb{N}$  and a  $(1+n)$ -ary term  $q$  and unary terms  $v_1, \dots, v_n, w_1, \dots, w_n$  such that the identities

$$x = q(x, v_1(0), \dots, v_n(0)),$$

$$0 = q(x, w_1(0), \dots, w_n(0))$$

hold in  $\mathcal{V}$ .

**Example 3.** For the variety of Boolean algebras, we can set  $n = 1$ ,  $v_1(x) = x'$ ,  $w_1(x) = x$  and  $q(x, y) = x \wedge y$ . Then  $q(x, v_1(0)) = x \wedge 0' = x$ , and  $q(x, w_1(0)) = x \wedge 0 = 0$ .

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