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ON MONADIC QUANTALE ALGEBRAS: BASIC PROPERTIES AND REPRESENTATION THEOREMS

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Abstract

Motivated by the concept of quantifier (in the sense of P. Halmos) on different algebraic structures (Boolean algebras, Heyting algebras, MV-algebras, orthomodular lattices, bounded distributive lattices) and the resulting notion of monadic algebra, the paper introduces the concept of a monadic quantale algebra, considers its properties and provides several representation theorems for the new structures.

Keywords: m-semilattice, $\bigvee$-lattice, quantale, quantale module, topological system, tropological system, quantale algebra, quantaloid, quantale algebroid, quantifier, monadic quantale algebra, Girard quantale, Q-equivalence relation, Ω-valued set, GL-monoid, commutative integral cl-monoid.

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